Compactness of Fuzzy Logics

Mirko Navara
Center for Machine Perception, Faculty of Electrical Engineering
Czech Technical University, Technická 2
166 27 Praha, Czech Republic
navara@cmp.felk.cvut.cz

Ulrich Bodenhofer
Software Competence Center Hagenberg
Hauptstrasse 99
A-4232 Hagenberg, Austria
ulrich.bodenhofer@scch.at

Abstract

This contribution is concerned with the compactness of fuzzy logics, where we concentrate on the three standard fuzzy logics—Łukasiewicz logic, Gödel logic, and product logic.

Keywords: fuzzy logic, many-valued logic, residuum, satisfiability, compactness of a logic.

The necessity to describe vagueness of data lead to the development of many-valued or fuzzy logics which use a larger scale of truth values, usually the real interval $[0,1]$. It is natural to ask to which extent the theorems known from classical logic hold also in this generalized context. One of the basic properties of classical logic is its compactness: If we have a set Γ of formulas such that each of its finite subsets is satisfiable, then Γ is satisfiable (i.e., all formulas in Γ can be simultaneously satisfied for some evaluation). We try to extend this result to fuzzy logics. The problem becomes more complex, because we have a continuum of truth values $[0,1]$ instead of two truth values of classical logic. Thus we have different types of satisfiability. While Łukasiewicz logic is known to satisfy the compactness property, we prove that this is not the case for Gödel and product logic. Nevertheless, some partial positive answers are obtained for these logics, too.

Following [3,4,11], we deal here with fuzzy logics which have the real interval $[0,1]$ as the set of truth values and the following basic connectives:

- the nullary false statement $\mathbf{0}$, interpreted by 0,
- the binary conjunction $\wedge$, interpreted by a *triangular norm* $T\colon [0,1]^2 \to [0,1]$, i.e., a commutative, associative, non-decreasing operation with neutral element 1,
- the binary implication $\to$, interpreted by the *residuum* R of T, i.e.,

$$R(x,y) = \sup\,\{z \in [0,1] : T(x,z) \le y\}\ .$$

In this approach, the semantics of the fuzzy logic under consideration is fully determined by the choice of the triangular norm T.

We start from a nonempty countable set A of *atomic symbols* and we define the class of *well-formed formulas* in a fuzzy logic (*formulas* for short) inductively as follows:

(i) Each atomic symbol p is a formula.

(ii) If $\Box$ is an n-ary connective and $\varphi_1, \dots, \varphi_n$ are formulas, then $\Box(\varphi_1, \dots, \varphi_n)$ is a formula.

For each function e which assigns a truth value to each atomic formula there exists always a unique *natural extension* of e to an *evaluation* $\overline{e}$ which, for each atomic symbol p, for each n-ary connective $\Box$, its interpretation $\mathrm{Meaning}_{\Box}$ and for all formulas $\varphi_1, \dots, \varphi_n$, is obtained by induction in the following canonical way:

$$\overline{e}(p) = e(p) ,$$
$$\overline{e}(\Box(\varphi_1, \dots, \varphi_n)) = \mathrm{Meaning}_{\Box}(\overline{e}(\varphi_1), \dots, \overline{e}(\varphi_n)) .$$

The three basic triangular norms lead to the following three main examples of fuzzy logics:

- For the triangular norm $T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$ we obtain *Łukasiewicz logic*.
- For the minimum $T_{\mathbf{G}}(x, y) = \min(x, y)$ we obtain *Gödel logic*.
- For the algebraic product $T_{\mathbf{P}}(x, y) = x \cdot y$ we obtain *product logic*.

For additional information on these logics we refer to [3,4,11]. Their detailed study and the proofs of completeness can be found in [4,6].

Using the basic logical connectives $\wedge, \rightarrow$ and $\mathbf{0}$, we can define derived logical connectives. Negation $\neg$ is defined as

$$\neg\varphi = \varphi \rightarrow \mathbf{0} .$$

Its interpretation is the fuzzy negation N given by

$$N(x) = R(x, 0) = \sup\{z \in [0,1] : T(x, z) \leq 0\} .$$

In Łukasiewicz logic this leads to standard fuzzy negation $N_{\mathbf{S}}(x) = 1 - x$, in Gödel and product logic we obtain Gödel negation

$$N_{\mathbf{G}}(x) = \begin{cases} 1 & \text{if } x = 0 , \\ 0 & \text{if } x > 0 . \end{cases}$$

For a conjunction with n equal arguments φ, we use the abbreviation $\bigwedge^n \varphi$. The inductive definition is:

$$\bigwedge^1 \varphi = \varphi ,$$
$$\bigwedge^{n+1} \varphi = \varphi \wedge \bigwedge^n \varphi , \quad n \in \mathbb{N} .$$

(We use the notation $\mathbb{N}$, resp. $\mathbb{N}_2$, for the set of positive integers, resp. all integers greater than 1.)

Definition 1. For a set Γ of formulas and $K \subseteq [0,1]$, we say that Γ is *K-satisfiable* if there exists an evaluation $\overline{e}$ such that we have $\overline{e}(\varphi) \in K$ for all $\varphi \in \Gamma$. The set Γ is said to be *finitely K-satisfiable* if each finite subset of Γ is K-satisfiable.

Obviously, K-satisfiability implies finite K-satisfiability. The reverse implication holds in classical logic, as well as in some fuzzy logics provided that K is closed. This property is called compactness of a logic.

Definition 2. A logic satisfies the *compactness property* if, for each closed subset K of $[0,1]$, K-satisfiability is equivalent to finite K-satisfiability.

In Łukasiewicz logic we obtain the interpretation $R_{\mathbf{L}}$ of the implication defined by $R_{\mathbf{L}}(x, y) = \min(1 - x + y, 1)$. In this case the following theorem holds:

Theorem 3. [4] *Łukasiewicz logic has the compactness property.*

Proof. Let Γ be a set of formulas. For each $\varphi \in \Gamma$, the mapping $H_\varphi \colon [0,1]^A \rightarrow [0,1]$ defined by $H_\varphi(e) = \overline{e}(\varphi)$ is continuous. The preimages $(H_\varphi^{-1}(K))_{\varphi \in \Gamma}$ are closed subsets of the compact $[0,1]^A$. This collection is centered (i.e., each finite subset has a nonempty intersection) because of finite K-satisfiability. Hence the intersection $\bigcap_{\varphi \in \Gamma} H_\varphi^{-1}(K)$ is nonempty; each of its elements is an evaluation that makes Γ K-satisfiable. $\Box$

The compactness property can be proved similarly also in other fuzzy logics in which all operations are interpreted by continuous functions. In particular, this applies to so-called *S-fuzzy logics*. These logics were introduced (under a different name) in [2] and investigated in detail in [7–9]. In S-fuzzy logics, the basic connectives are negation $\neg$, interpreted by standard fuzzy negation $N_{\mathbf{S}}(x) = 1 - x$, and conjunction $\wedge$, interpreted by a continuous triangular norm. Implication $\rightarrow$ in an S-fuzzy logic is a derived connective

$$\varphi \rightarrow \psi = \neg(\varphi \wedge \neg\psi) .$$

The compactness property of S-fuzzy logics is proved in [2, Th. 3.3].

The same argument does not work in Gödel and product logic where the residua $R_{\mathbf{G}}$ and $R_{\mathbf{P}}$ interpreting the implication are defined by

$$R_{\mathbf{G}}(x,y) = \begin{cases} 1 & \text{if } x \le y \ , \\ y & \text{otherwise} \ , \end{cases}$$

$$R_{\mathbf{P}}(x,y) = \begin{cases} 1 & \text{if } x \le y \ , \\ \dfrac{y}{x} & \text{otherwise} \ . \end{cases}$$

The residuum $R_{\mathbf{G}}$ is not continuous in the points (x,x), $0 \le x < 1$, $R_{\mathbf{P}}$ has a discontinuity in $(0,0)$. Still some partial positive results can be obtained. For instance, in Gödel logic, finite $\{1\}$-satisfiability is equivalent to $\{1\}$-satisfiability. This weakened form of compactness was proved in [4]. In a modified form it can be derived from [1]; although the logic and notions studied there do not coincide exactly with our approach, the paper covers this particular result. It was generalized by Hájek to the following form [5]:

Theorem 4. *Let K be a closed subset of $[0,1]$ such that $1 \in K$, $0 \notin K$. Then in Gödel logic, as well as in product logic, finite K-satisfiability implies K-satisfiability.*

Another partial positive result for Gödel logic is the following (again, see [1] for a related result):

Proposition 5. *Gödel logic with finitely many atomic symbols has the compactness property.*

Proof. With finitely many atomic symbols, we can form infinitely many formulas, but only finitely many of them are semantically different. (We call two formulas semantically different if there is an evaluation attaining different values on them.) As satisfiability depends only on the semantics, we get the compactness property trivially. □

However, the compactness property of Gödel logic does not hold in general:

Theorem 6. *Gödel logic with an infinite set of atomic symbols does not satisfy the compactness property.*

Proof. Let p_n, $n \in \mathbb{N}$, be atomic symbols. We take

$$\Gamma = \{p_{n+1} \to p_n : n \in \mathbb{N}\} \ ,$$

$$K = \{0\} \cup \{1/n : n \in \mathbb{N}_2\} \ .$$

Let $\overline{e}$ be an evaluation such that $\overline{e}(p_{n+1} \to p_n) = x$, where $x \in K$. As $x < 1$, this is possible only if

$$\overline{e}(p_n) = x \ , \qquad \overline{e}(p_{n+1}) > x \ .$$

For each $m \in \mathbb{N}$, the set

$$\Gamma_m = \{p_{n+1} \to p_n : n = 1, \ldots, m\} \subseteq \Gamma$$

is K-satisfiable. Indeed, if we take

$$\overline{e}(p_n) = \frac{1}{m+2-n} \ , \qquad n = 1, \ldots, m+1 \ ,$$

we obtain

$$\overline{e}(p_{n+1} \to p_n) = \frac{1}{m+2-n} \in K \ .$$

Each finite subset of Γ is contained in Γ_m for some $m \in \mathbb{N}$, so Γ is finitely K-satisfiable.

It remains to prove that Γ is not K-satisfiable. Suppose that $\overline{e}$ is an evaluation which maps all formulas from Γ into K. This implies that the sequence $(\overline{e}(p_n))_{n \in \mathbb{N}}$ is strictly increasing. We get a contradiction, because $\overline{e}(p_2) \in K \backslash \{0\}$ and there are only finitely many (exactly $1/\overline{e}(p_2) - 1$) elements of K greater than $\overline{e}(p_2)$. Thus such an evaluation does not exist. □

Also in product logic, we do not obtain compactness in general.

Theorem 7. *Product logic does not satisfy the compactness property.*

Proof. (We only outline the basic ideas to give the reader an overview—to show which set of formulas and which closed set of truth values are used. Their properties are not proved here. The complete proof exceeds the scope of this paper; it can be found in [10].)

We define a set

$$M = \left\{ i - \frac{1}{j} : \quad i, j \in \mathbb{N}_2 ,\ j \le i \right\} \ .$$

The collection of sets $\{\frac{1}{n} M : n \in \mathbb{N}\}$ has the following intersections:

1. $\bigcap_{n \in F} \frac{1}{n} M \neq \emptyset$ for each finite subset $F \subseteq \mathbb{N}$,

2. $\bigcap_{n \in \mathbb{N}} \frac{1}{n} M = \emptyset$.

We define the set

$$K = \exp(-M) \cup \{0\} = \{\exp(-z) : z \in M\} \cup \{0\} .$$

(We use here exp to denote the exponential function, resp. its extension to sets of reals.) The set K is a closed subset of $[0, 1]$.

We take an atomic symbol p. For each $n \in \mathbb{N}$, we define the formula φ_n by

$$\varphi_n = \neg\neg p \to \bigwedge^n p .$$

The function $H_{\varphi_n} : e(p) \mapsto \overline{e}(\varphi_n)$ interpreting the formula φ_n is

$$\begin{aligned} H_{\varphi_n}(x) &= R_{\mathbf{P}}(N_{\mathbf{G}}(N_{\mathbf{G}}(x)), x^n) \\ &= \begin{cases} x^n & \text{if} \quad x > 0 , \\ 1 & \text{if} \quad x = 0 . \end{cases} \end{aligned}$$

The set $\Gamma = \{\varphi_n : n \in \mathbb{N}\}$ is finitely K-satisfiable, but not K-satisfiable. This can be proved by using the preimages

$$H_{\varphi_n}^{-1}(K) = \left\{ \exp\left(-\frac{z}{n}\right) : z \in M \right\} = \exp\left(-\frac{1}{n} M\right)$$

and the above properties of the collection of sets $\{\frac{1}{n} M : n \in \mathbb{N}\}$. □

Although we do not have the compactness property in Gödel and product logic, it still holds for some special forms of the set K. Also in some other fuzzy logics the questions of compactness were not clarified yet. These problems may be a subject for further research.

Acknowledgements

Mirko Navara is supported by the Czech Ministry of Education under Research Programme MSM 212300013 Decision and Control for Industry.

Ulrich Bodenhofer is working in the framework of the *Kplus Competence Center Program* which is funded by the Austrian Government, the Province of Upper Austria, and the Chamber of Commerce of Upper Austria.

References

[1] M. Baaz and R. Zach. Compact propositional Gödel logics. In *Proc. 28th Int. Symp. on Multiple Valued Logic*. IEEE Computer Society Press, Los Alamitos, CA, 1998.

[2] D. Butnariu, E. P. Klement, and S. Zafrany. On triangular norm-based propositional fuzzy logics. *Fuzzy Sets and Systems*, 69:241–255, 1995.

[3] S. Gottwald. *A Treatise on Many-Valued Logics*. Studies in Logic and Computation. Research Studies Press, Baldock, 2001.

[4] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer Academic Publishers, Dordrecht, 1998.

[5] P. Hájek. Personal communication. August 2000.

[6] P. Hájek, L. Godo, and F. Esteva. A complete many-valued logic with product conjunction. *Arch. Math. Logic*, 35:191–208, 1996.

[7] J. Hekrdla, E. P. Klement, and M. Navara. Two approaches to fuzzy propositional logics. (to appear).

[8] E. P. Klement and M. Navara. A survey of different triangular norm-based fuzzy logics. *Fuzzy Sets and Systems*, 101:241–251, 1999.

[9] M. Navara. Satisfiability in fuzzy logics. *Neural Network World*, 10(5):845–858, 2000.

[10] M. Navara. Product logic is not compact. Technical Report CTU-CMP-2001-09, Center of Machine Perception, Czech Technical University, Prague, Czech Republic, 2001.

[11] E. Turunen. *Mathematics Behind Fuzzy Logic*. Advances in Soft Computing. Physica-Verlag, Heidelberg, 1999.