Modelling Linguistic Expressions Using Fuzzy Relations

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Abstract— The concept of an image of a fuzzy set under a fuzzy relation has proved to be a very powerful tool in fuzzy set theoretical applications. In this paper, we explain how it can be used to model linguistic expressions. For the representation of expressions, such as "at least middle-aged", "brighter than average", we will use fuzzy ordering relations, while resemblance relations will be suitable to model linguistic terms, such as "more or less expensive" and "very tall." We will show how these representations can be smoothly integrated in approximate reasoning schemes using the compositional rule of inference.

1 Introduction

The ability to model expressions from natural language plays a fundamental role in the success story of fuzzy set theory for practical applications. An important part within this framework is devoted to the representation of linguistic modifiers, i.e. linguistic expressions by which other expressions are modified. Generally, a linguistic term is modelled by a fuzzy set and a linguistic modifier, therefore, by an operation that transforms a fuzzy set into another. In the literature, plenty of this kind of operators have been suggested already [11]. However, these operators in general lack significant inherent meaning: They are tools whose definition is only dictated by the technical criterion that they should transform some fuzzy sets into (approximations of) some other fuzzy sets, but they have no further meaning of their own.

In contrast to these traditional representations, in [3] and [6], two new approaches are proposed that construct representations for linguistic modifiers based on an underlying semantics. As we shall explain in detail, this is done by taking mutual relationships between objects of the universe into account. In [3], a fuzzy ordering relation is used to model ordering-based modifiers (e.g. at least), while in [6] a resemblance relation is used to represent weakening modifiers (e.g. more or less) and intensifying modifiers (e.g. very). In this paper, we join these two approaches in a general framework of images under fuzzy relations. From the generalization, also a representation for comparing linguistic expressions, such as "greater than average", arises, and a link between the images and the fuzzy concepts *up-set* and *appropriateness* is established.

After recalling some preliminaries (Section 2), we give the definitions of the key tools to our approach, i.e. images of fuzzy sets under fuzzy relations. We state the properties that will be useful for the application we want to deal with, namely the representation of linguistic expressions (Section 3). To establish such representations, first we focus on ordering relations (Section 4). Then we come to resemblance relations, i.e. relations that model approximate equality (Section 5). Finally, we show how the developed representations fit into approximate reasoning schemes (Section 6).

2 Preliminaries

Throughout this paper, let X denote a universe. $\mathcal{F}(X)$ is the class of all fuzzy sets on X, while $\mathcal{P}(X)$ denotes as usual the class of all crisp subsets of X.

For A and B fuzzy sets on X, the union and intersection are the fuzzy sets on X defined as follows (for x in X):

$$(A \cup B)(x) = \max(A(x), B(x))$$
$$(A \cap B)(x) = \min(A(x), B(x))$$

If A and B represent linguistic terms, then $A \cup B$ and $A \cap B$ are usually interpreted as "A or B" and "A and B", respectively. Furthermore,

$$A \subseteq B \Leftrightarrow (\forall x \in X)(A(x) \leq B(x)).$$

For α in $[0,1]$, the so-called α-cut of A is defined as

$$[A]_\alpha = \{x \in X \mid A(x) \geq \alpha\}.$$

We say that A is normalized if and only if $[A]_1 \neq \emptyset$. The height of A is given by

$$\text{hgt}(A) = \sup\{A(x) \mid x \in X\}.$$

A fuzzy relation R on X is a fuzzy set on $X \times X$. For y in X, the R-foreset and the R-afterset of y, denoted by Ry and yR, respectively, are fuzzy sets on X defined as

$$(Ry)(x) = R(x, y),$$
$$(yR)(x) = R(y, x).$$

Note that $(R^{-1})y = yR$ and $y(R^{-1}) = Ry$. A fuzzy relation R is called reflexive if and only if $R(x, x) = 1$ for all x in X.

In the remaining paper, let T denote a left-continuous triangular norm (t-norm for short) with a unique residual implication [8]

$$\vec{T}(x, y) = \sup\{\gamma \in [0, 1] \mid T(\gamma, x) \leq y\}.$$

In particular, the following properties of T and $\vec{T}$ will be important in this paper (for arbitrary x, x_1, x_2, y, y_1, y_2 in $[0, 1]$):

$$T(1, x) = T(x, 1) = x \quad (1)$$
$$\vec{T}(1, x) = x \quad (2)$$
$$\vec{T}(x, y) = 1 \Leftrightarrow x \leq y \quad (3)$$

$$x_1 \leq x_2 \Rightarrow T(x_1, y) \leq T(x_2, y) \quad (4)$$
$$y_1 \leq y_2 \Rightarrow T(x, y_1) \leq T(x, y_2) \quad (5)$$
$$x_1 \leq x_2 \Rightarrow \vec{T}(x_1, y) \geq \vec{T}(x_2, y) \quad (6)$$
$$y_1 \leq y_2 \Rightarrow \vec{T}(x, y_1) \leq \vec{T}(x, y_2) \quad (7)$$

Let us recall that the Lukasiewicz t-norm $T_{\mathbf{L}}$ and the Lukasiewicz implication $\vec{T}_{\mathbf{L}}$ are defined as

$$T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0),$$
$$\vec{T}_{\mathbf{L}}(x, y) = \min(1 - x + y, 1).$$

3 Basic Concepts

Definition 1 (Degree of Overlapping). For A and B in $\mathcal{F}(X)$, the degree of overlapping of A and B is defined by

$$\mathsf{OVERL}(A, B) = \sup_{x \in X} T(A(x), B(x)).$$

Definition 2 (Degree of Inclusion). [2] For A and B in $\mathcal{F}(X)$, the degree to which A is included in B is defined as

$$\mathsf{INCL}(A, B) = \inf_{x \in X} \vec{T}(A(x), B(x)).$$

Note that $A \subseteq B$ if and only if $\mathsf{INCL}(A, B) = 1$ [4].

Definition 3 (Images). For R a fuzzy relation on X and A a fuzzy set on X, the images $R(A)$, $R^{\triangle}(A)$ and $R^{\nabla}(A)$ are fuzzy sets on X defined as follows (for all $y \in X$):

$$R(A)(y) = \sup_{x \in X} T(A(x), R(x, y)) = \mathsf{OVERL}(A, Ry)$$
$$R^{\triangle}(A)(y) = \inf_{x \in X} \vec{T}(A(x), R(x, y)) = \mathsf{INCL}(A, Ry)$$
$$R^{\nabla}(A)(y) = \inf_{x \in X} \vec{T}(R(y, x), A(x)) = \mathsf{INCL}(yR, A)$$

$R(A)$ is called the direct or the full image of A under R [8, 10], while $R^{\triangle}(A)$ and $R^{\nabla}(A)$ are closely related to the subdirect image $R^{\triangleleft}(A)$ and the superdirect image $R^{\triangleright}(A)$ of A under R [10]. More precisely,

$$R^{\triangleleft}(A) = R(A) \cap R^{\triangle}(A),$$
$$R^{\triangleright}(A) = R(A) \cap (R^{-1})^{\nabla}(A).$$

Proposition 1. *If R is a reflexive relation on X then the following holds for all A in $\mathcal{F}(X)$:*

$$R^{\nabla}(A) \subseteq A \subseteq R(A)$$

Proof. For all y in X:

$$\begin{aligned} R^{\nabla}(A)(y) &\leq \vec{T}(R(y, y), A(y)) \\ &\leq A(y) \\ &\leq T(R(y, y), A(y)) \\ &\leq R(A)(y) \end{aligned}$$ □

The following two propositions arise from the monotonic behaviour of supremum, infimum, t-norms, and residual implications.

Proposition 2. *If R_1 and R_2 are fuzzy relations on X such that $R_1 \subseteq R_2$ then for all A in $\mathcal{F}(X)$:*

(a) $R_1(A) \subseteq R_2(A)$
(b) $R_1^{\nabla}(A) \supseteq R_2^{\nabla}(A)$

Proposition 3. *If A and B are fuzzy sets on X such that $A \subseteq B$ then we have, for all R in $\mathcal{F}(X \times X)$,*

$$R^{\triangle}(A) \supseteq R^{\triangle}(B).$$

Proposition 4. *If A is a normalized fuzzy set on X then for $R \in \mathcal{F}(X \times X)$:*

$$R^{\triangle}(A) \subseteq R(A)$$

Proof. Since A is normalized, there is an x_0 in X such that $A(x_0) = 1$. Then we obtain, for y in X:

$$\begin{aligned} R^{\triangle}(A)(y) &\leq \vec{T}(A(x_0), R(x_0, y)) \\ &\leq R(x_0, y) \\ &\leq T(A(x_0), R(x_0, y)) \\ &\leq R(A)(y) \end{aligned}$$ □

4 Using Orderings

4.1 Ordering relations

We begin this section by recalling several concepts of crisp and fuzzy orderings.

Definition 4 (Ordering). A relation L on X is called an ordering relation (ordering for short) if and only if, for all x, y, and z in X:

(O.1) $L(x,x) = 1$ (refl.)
(O.2) $L(x,y) = 1 \wedge L(y,x) = 1 \Rightarrow x = y$ (anti-symm.)
(O.3) $L(x,y) = 1 \wedge L(y,z) = 1 \Rightarrow L(x,z) = 1$ (trans.)

Since L is a relation on X, we can use the concepts of L-foreset and L-afterset as defined in Section 2. If L represents the relation "is smaller than or equal to" on X then, for each x in X,

$$Lx = \{y \in X \mid L(y,x) = 1\}$$

is the set of objects that are smaller or equal to x. Furthermore, the inverse relation L^{-1} — also an ordering — represents "is greater than or equal to".

Definition 5 (Strict ordering). A relation S on X is called a strict ordering on X if and only if, for all x, y, and z in X:

(S.1) $S(x,x) = 0$ (anti-refl.)
(S.2) $S(x,y) = 1 \wedge S(y,z) = 1 \Rightarrow S(x,z) = 1$ (trans.)

If S is a strict ordering on X then the relation L defined by

$$L(x,y) = 1 \Leftrightarrow \big(S(x,y) = 1 \vee x = y\big) \tag{8}$$

is an ordering on X.

In [3], it is argued that, if there is a notion of fuzzy similarity or indistinguishability in the universe X, an ordering on X should take this into account. Indeed, although it can be argued that a height of 1.801 (meter) is not smaller than 1.800, the human eye can not always make the distinction. Therefore, in some contexts or applications, it is not wise to exclude 1.801 completely from the set "smaller than 1.800." In other words, 1.801 should be considered as "(more or less) smaller than 1.800" to a degree greater than 0. For this purpose, the concept of a fuzzy ordering, which takes the strong connection between similarity and ordering into account, is defined. First, we recall the definition of fuzzy equivalence which is the common concept for modelling similarity.

Definition 6 (Fuzzy T-equivalence). A fuzzy relation E on X is called a fuzzy T-equivalence on X if and only if, for all x, y, and z in X:

(E.1) $E(x,x) = 1$ (refl.)
(E.2) $E(x,y) = E(y,x)$ (symm.)
(E.3) $T(E(x,y), E(y,z)) \leq E(x,z)$ (T-trans.)

Definition 7 (Fuzzy T-E-ordering). Let E be a fuzzy T-equivalence on X. A fuzzy relation F on X is called a fuzzy T-E-ordering on X if and only if, for all x, y, and z in X:

(L.1) $E(x,y) \leq F(x,y)$ (E-refl.)
(L.2) $T(F(x,y), F(y,x)) \leq E(x,y)$ (T-E-antisymm.)
(L.3) $T(F(x,y), F(y,z)) \leq F(x,z)$ (T-trans.)

It can be easily verified that this definition includes crisp orderings on X (for E the crisp equality on X and T an arbitrary t-norm). In the following, we shall use the term "fuzzy ordering", thereby tacitly assuming the existence of a suitable T and E. Note that the inverse of a fuzzy ordering is also a fuzzy ordering.

4.2 Ordering-based modifiers

Let L be a crisp ordering on X denoting "is smaller than or equal to," and x a crisp value in X. For y in X, we say that "y is at least x" if and only if y is greater than or equal to x. Hence the set "at least x" can be defined as $(L^{-1})x$. This notion can be generalized to a crisp subset P of X as follows:

$y \in$ at least P
$\Leftrightarrow$ y is greater than or equal to **some** element of P
$\Leftrightarrow (\exists z \in X)(z \in P \wedge L^{-1}(y,z) = 1)$
$\Leftrightarrow (\exists z \in X)(z \in P \wedge z \in Ly)$

In other words, y belongs to "at least P" if and only if the intersection of P and Ly is not empty. Note that, in this definition, an element y is in at least x if and only if $y \in$ at least $\{x\}$. Since Ly and P are both crisp, this is equivalent to stating that y belongs to at least P if and only if the degree of overlapping $\mathsf{OVERL}(P, Ly)$ is 1. In [3] and [4], this underlying meaning is generalized to a fuzzy set A on X as follows:

$$\text{at least } A\ (y) = \mathsf{OVERL}(A, Ly) = L(A)(y),$$

where $L(A)$ denotes the direct image of A under the relation L as defined in Definition 3.

If we also want to take the fuzzy similarity into account, we can use a fuzzy ordering F on X that represents "is smaller than or equal to":

$$\text{(more or less) at least } A\ (y) = \mathsf{OVERL}(A, Fy) = F(A)(y)$$

Using the inverse orderings, "at most A" can be analogously represented by $L^{-1}(A)$, while $F^{-1}(A)$ is suitable for "(more or less) at most A".

Example 1. Figure 1 depicts fuzzy sets labelled open, half-open, almost closed, and closed of a variable "valve opening of a fermenter" (membership functions taken from [1]). Using the common orderings $\leq$ and $\geq$ of real numbers, the representation of the linguistic expression

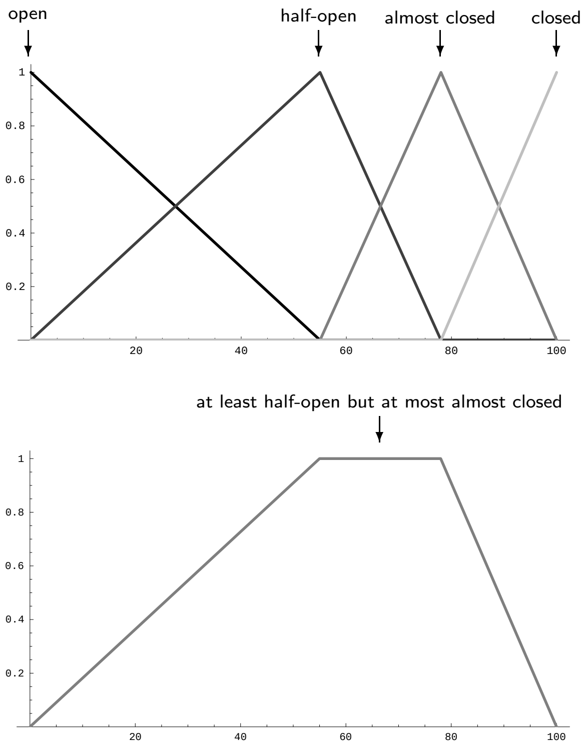

Figure 1: Valve opening

"at least half-open but at most almost closed" can be constructed. More specifically,

$$\leq (\text{half open}) \cap \geq (\text{almost closed}).$$

Since the ordering relations are crisp, the choice of a particular t-norm does not influence the result.

4.3 Ordering-based comparisons

Now let S be a strict ordering on X representing "is smaller than" and let x be a crisp value in X. For y in X, we say that "y is greater than x" if and only if $S^{-1}(y,x) = 1$. More generally, for P in $\mathcal{P}(X)$, we can define:

$$\begin{aligned}
& y \in \text{greater than } P \\
& \Leftrightarrow y \text{ is greater than } \textbf{all} \text{ elements of } P \\
& \Leftrightarrow (\forall x \in X)(x \in P \Rightarrow S^{-1}(y,x) = 1) \\
& \Leftrightarrow (\forall x \in X)(x \in P \Rightarrow x \in Sy)
\end{aligned}$$

This means that y belongs to "greater than P" if and only if P is included in Sy. Again this can be generalized to a fuzzy set A on X:

$$\text{greater than } A\ (y) = \mathsf{INCL}(A, Sy) = S^{\triangle}(A)(y)$$

Similarly, we can use $(S^{-1})^{\triangle}(A)$ to represent the fuzzy set "smaller than A".

In applications, the ordering often has a more specific interpretation than "is greater than" or "is smaller than"; the representation presented here can be used to model a variety of ordering-based comparing linguistic expressions, e.g. in a universe of ages we can construct the sets "older than middle-aged", "younger than about 20", while in a universe of IQs we can use the representation to model "brighter than average", etc.

If A represents the concept "tall" in a universe X of heights, then the following property supports the intuitive idea that a man who is "tall or taller than tall" can be called "at least tall".

Proposition 5. *Let S be a strict order on X and let L be the associated ordering defined by Formula (8). If A is a normalized fuzzy set on X then*

$$A \cup S^{\triangle}(A) \subseteq L(A).$$

Proof. A is normalized, hence $S^{\triangle}(A) \subseteq S(A)$ (see Proposition 4). Since S is a subset of L, $S(A) \subseteq L(A)$ (see Proposition 2). Therefore $S^{\triangle}(A) \subseteq L(A)$. L is reflexive, so combined with Proposition 1 this yields the stated property. □

The opposite inclusion does not always hold intuitively. This is illustrated by the following example.

Example 2. Suppose that the mean score of a test is 14. The fuzzy set "average" in the universe of scores $[0, 20]$ can be represented by the triangular fuzzy set $A = (12, 14, 16)$ shown in Figure 2. Using the Lukasiewicz t-norm and implication, respectively, the membership functions for "at least average" and "more than average" in Figure 2 correspond to the fuzzy sets $\leq (A)$ and $< (A)$, with $\leq$ and $<$ denoting the common ordering and strict ordering on real numbers. A score of 15 is (intuitively) "at least average" to degree 1. However, 15 is not called "average" to degree 1, neither can it be called "more than average" to degree 1, since a lot of scores that are greater than 15 are still considered to be average to some degree (e.g. 15.1, 15.5, 15.9, ...).

If A is not normalized some intuitive problems may occur. In particular, we have "smaller than $\emptyset = X$". This problem can be solved by representing "smaller than A" by the subdirect image $(S^{-1})^{\triangleleft}(A)$ instead of $(S^{-1})^{\triangle}(A)$, thus making sure that "smaller than A" is a subset of "at most A".

Remark 1. Using the representation for "greater than A" introduced in this section and the concept of degree of inclusion defined in Definition 2, we can construct an ordering on fuzzy sets (for two fuzzy sets $A, B \in \mathcal{F}(X)$):

$$B \text{ is greater than } A \text{ to degree } \mathsf{INCL}(B, \text{greater than } A)$$

Note that, if we also want to be able to compare fuzzy sets with the empty set, some extra precautions have to be taken to avoid the counter-intuitive result that $\emptyset$ is greater than all fuzzy sets to degree 1.

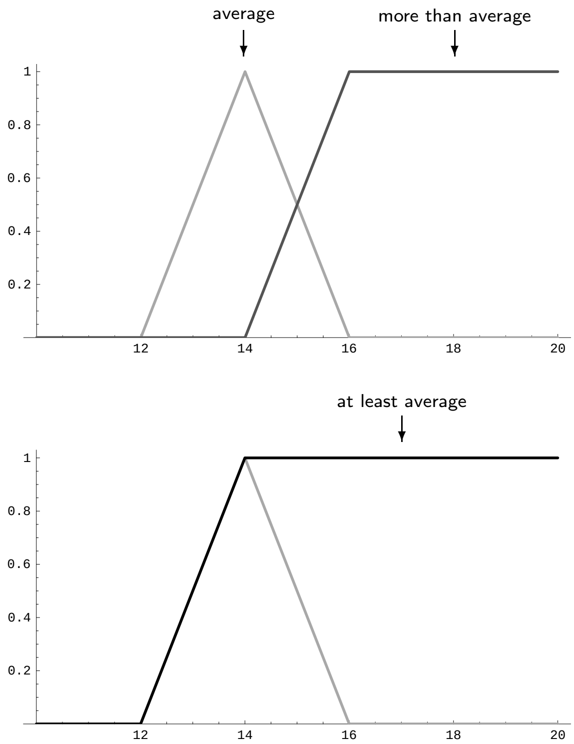

Figure 2: Score of a test

4.4 Up-set

The only kind of image (see Definition 3) of a fuzzy set A under an ordering L that still needs to be discussed is $L^{\nabla}(A)$. In this section, we show that this image is related to a fuzzy concept of up-set. First, we recall the definition of up-set in the crisp case [5].

Definition 8 (Up-set I). Let L be an ordering on X and P in $\mathcal{P}(X)$. P is an up-set in (X, L) if and only if

$$(\forall y \in P)(\forall x \in X)(L(y, x) = 1 \Rightarrow x \in P)$$

This notion can be fuzzified as follows:

Definition 9 (Up-set II). Let F be a fuzzy ordering on X. For A in $\mathcal{F}(X)$, the degree to which A is an up-set in (X, F) is defined by:

$$\begin{aligned}\mathsf{Up}(A) &= \inf_{y \in X} \vec{T}\big(A(y), \inf_{x \in X} \vec{T}(F(y,x), A(x))\big)\\ &= \mathsf{INCL}(A, F^{\nabla}(A))\end{aligned}$$

A is called a *true up-set* in (X, F) if and only if $\mathsf{Up}(A) = 1$.

Proposition 6. *A is a true up-set in (X, F) if and only if $A = F^{\nabla}(A)$.*

Proof. A is a true up-set in (X, F) if and only if $A \subseteq F^{\nabla}(A)$. Since $F^{\nabla}(A) \subseteq A$ always holds for a reflexive relation F (see Proposition 1), this is equivalent to $A = F^{\nabla}(A)$. □

5 Using Resemblance Relations

5.1 Resemblance relations

In this section, we will study the images defined in Definition 3 for the case that the relation R models approximate equality. The intuitive meaning of the notion of *approximate equality* involves *reflexivity* (an object x is approximately equal to itself to degree 1) and *symmetry* (x is approximately equal to y to the same degree as y is approximately equal to x). However, assuming that approximate equality should be T-transitive like fuzzy T-equivalences, may lead to counter-intuitive results. To illustrate this, the following example is given in [7]:

Example 3. In every-day life we usually do not feel a difference in temperature between 0° and 1°, neither between 1° and 2°, between 35° and 36°, etc. For us, 0° and 1° are certainly approximately equal, and so are 1° and 2°, and 35° and 36°, etc. To formalize this, consider a universe X of temperatures and a T-equivalence relation E on X used to represent "*approximately equal*". We would thus expect,

$$E(k, k+1) = 1 \quad \text{for every } k \text{ in } \mathbb{N}.$$

By induction, it is easy to show that, for every k and n in $\mathbb{N}$, $E(k, k+n) = 1$. This means, in turn, that all temperatures are approximately equal to the degree 1 — obviously, a completely counter-intuitive result.

The above discussion leads us to defining resemblance relations: These relations are reflexive and symmetric; T-transitivity, however, is replaced by a pseudo-metric-based criterion.

Definition 10 (Pseudo-metric). An $\mathfrak{M}^2 \to \mathbb{R}_0^+$ mapping d is called a pseudo-metric on $\mathfrak{M}$ if and only if, for every x, y, and z in $\mathfrak{M}$:

(M.1) $d(x, x) = 0$
(M.2) $d(x, y) = d(y, x)$
(M.3) $d(x, y) + d(y, z) \geq d(x, z)$

$(\mathfrak{M}, d)$ is then called a pseudo-metric space.

Intuitively, we feel that the smaller the distance between two objects is, the more they are approximately equal. This is expressed by the axiom (R.3) in the following definition.

Definition 11 (Resemblance relation). A fuzzy relation R on X is called a resemblance relation on X if and only if there exists a pseudo-metric space $(\mathfrak{M}, d)$ and a $X \to \mathfrak{M}$ mapping g such that, for all x, y, z, and u in X:

(R.1) $E(x, x) = 1$
(R.2) $E(x, y) = E(y, x)$
(R.3) $d(g(x), g(y)) \preceq d(g(z), g(u)) \Rightarrow E(x, y) \geq E(z, u)$

If X is already equipped with a pseudo-metric, then g can be the identity mapping. Note that, for all y in X, Ey $(= yE)$ is the fuzzy set of all objects resembling to y.

5.2 Weakening modifiers

Let R denote a crisp resemblance relation on X, i.e. $R(x, y) = 1$ if and only if x resembles to y. Note that a person can be called "more or less adult" if he/she resembles in age to an adult. Therefore, in general, for P in $\mathcal{P}(X)$ and y in X, we can define:

$$\begin{aligned}&y \in \text{more or less } P\\ &\Leftrightarrow \text{some element of } P \text{ resembles to } y\\ &\Leftrightarrow (\exists x \in X)(x \in P \wedge R(x, y) = 1)\end{aligned}$$

In other words, y belongs to "more or less P" if and only if $\mathsf{OVERL}(P, Ry) = 1$. Using a resemblance relation E on X, this can be generalized to a fuzzy set A on X [6]:

$$\text{more or less } A\ (y) = \mathsf{OVERL}(A, Ey) = E(A)(y)$$

This means that y belongs to "more or less A" to the degree to which A and the fuzzy set of objects resembling to y overlap.

We can approach the representation of the term "roughly A" from two different sides. The first option is to use a second resemblance relation E_1 on X such that $E \subset E_1$ and to define:

$$\text{roughly } A\ (y) = \mathsf{OVERL}(A, E_1 y) = E_1(A)(y)$$

The second possibility is to define

$$\text{roughly } A\ (y) = E(E(A))(y)$$

Propositions 1 and 2 (a) guarantee that for both options

$$A \subseteq \text{more or less } A \subseteq \text{roughly } A \qquad (9)$$

which corresponds to the inclusive interpretation of these terms, which is often assumed in literature (see e.g. [12]).

Note that, for a reflexive and T-transitive fuzzy relation E,

$$E(E(A)) = E(A)$$

holds, which would make the second inclusion in Formula (9) an equality. This supports the statement posed in Section 5.1 that T-transitivity is not an appropriate requirement for relations modelling approximate equality. In the next example, a non-T-transitive resemblance relation E is chosen to model "more or less" and "roughly."

Example 4. Figure 3 depicts a membership function for a fuzzy set A representing the concept "tall" in the universe of heights of men. The membership functions for "more or less tall" and "roughly tall" in Figure 3 correspond to the fuzzy sets $E(A)$ and $E(E(A))$, respectively, with E defined as

$$E(x, y) = \min\big(1, \max(0, 1.5 - 0.1|x - y|)\big),$$

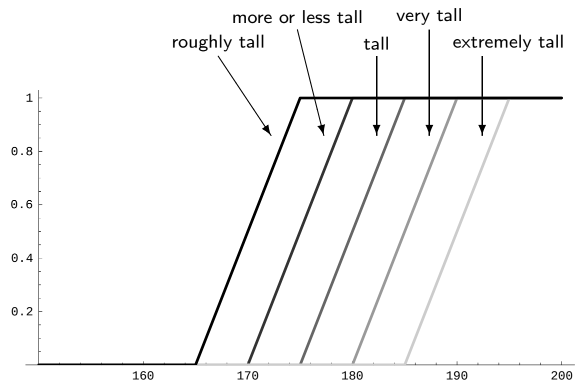

Figure 3: Weakening and intensifying modifiers

where $T_{\mathbf{L}}$ was chosen as t-norm. It can be shown that, for this t-norm,

$$E(E(A)) = E_1(A)$$

where the resemblance relation E_1 is defined by

$$E_1(x, y) = \min\big(1, \max(0, 2 - 0.1|x - y|)\big).$$

Hence, in this example, the two approaches for modelling "roughly" coincide.

5.3 Intensifying modifiers

For R a crisp concept of resemblance in X, P in $\mathcal{P}(X)$ and y in X, we can define:

$$\begin{aligned}&y \in \text{very } P\\ &\Leftrightarrow \text{all elements that } y \text{ resembles to, belong to } P\\ &\Leftrightarrow (\forall x \in X)(R(x, y) = 1 \Rightarrow x \in P)\end{aligned}$$

In natural language, the application of an intensifying modifier like "very" to a crisp concept usually has little or no meaning (what is understood by "very pregnant", "very rectangular", "very dead", etc.?). However, for a fuzzy concept, such as "beautiful", it is clear that a woman who resembles to **all** beautiful women must be "very beautiful". More formally, we define [6] (for A in $\mathcal{F}(X)$),

$$\text{very } A\ (y) = \mathsf{INCL}(yE, A) = E^{\nabla}(A)(y),$$

where E is a resemblance relation.

Analogous to the representation scheme for "roughly A", we have two choices to model "extremely A." Either we use a resemblance relation E_1 such that $E \subset E_1$, i.e.

$$\text{extremely } A\ (y) = \mathsf{INCL}(yE_1, A) = E_1^{\nabla}(A)(y),$$

or we define

$$\text{extremely } A\ (y) = E^{\nabla}(E^{\nabla}(A))(y).$$

Again Propositions 1 and 2 (b) guarantee that, for both options, the inclusive interpretation is respected:

$$\text{extremely } A \subseteq \text{very } A \subseteq A$$

Example 5. The membership functions for "very tall" and extremely tall" in Figure 3 correspond to the fuzzy sets $E^{\nabla}(A)$ and $E^{\nabla}(E^{\nabla}(A))$ respectively, with A and E defined as in Example 4. Again the $\vec{T}_{\mathbf{L}}$ implication was chosen. For this implication and the relation E_1 defined in Example 4, it also holds that

$$E^{\nabla}(E^{\nabla}(A)) = E_1^{\nabla}(A).$$

5.4 Appropriateness

The only kind of image of a fuzzy set A (according to Definition 3) under a resemblance relation E that we have not discussed so far is $E^{\triangle}(A)$. Reasoning in an analogous way as in the previous sections, we remark that, for y in X, $E^{\triangle}(A)(y)$ is the degree to which A is included in the fuzzy set of objects that resemble to y. Hence, we can use $E^{\triangle}(A)(y)$ to express something about the appropriateness of the term A for y. The degree to which A is an appropriate term for y cannot be greater than the degree to which y satisfies A (i.e. the degree to which y belongs to A). Therefore, we use $T(A(y), E^{\triangle}(A))$ to express the degree to which A is appropriate for y.

If a fuzzy set B on X represents a larger concept than A, i.e. $A \subseteq B$, and $A(y) = B(y)$, then B can never model a more appropriate term for y. Indeed Grice's [9] maxims of conversation indicate that out of two expressions with the same linguistic cost, the most informative one should be chosen. One can check that this is formally supported by Proposition 3.

6 Approximate Reasoning

The well-known compositional rule of inference (CRI) (see e.g. [13]) is based on the concept of direct image under a fuzzy relation. Therefore, representations of linguistic expressions based on the same image, as presented in previous sections, can be smoothly integrated in this approximate reasoning scheme. To demonstrate this, we will use the most popular form of the CRI.

Definition 12 (CRI). For $\mathbf{u}$ and $\mathbf{v}$ variables in X, A a fuzzy set on X, and R a fuzzy relation on X, the CRI dictates the following derivation:

$$\begin{array}{ll} \mathbf{u} \text{ is } A & (1) \\ \underline{\mathbf{u} \text{ and } \mathbf{v} \text{ are } R} & (2) \\ \mathbf{v} \text{ is } R(A) & (3) \end{array}$$

This means that, from the two pieces of knowledge (1) and (2), the new information stated in (3) can be derived.

The following two examples show how the representation for linguistic expressions fit into this framework.

Fred is a good student.	$\rightarrow$	$\mathbf{u}$ is A
Zdenek is as good or better.	$\rightarrow$	$\mathbf{u}$ and $\mathbf{v}$ are L
Zdenek it is at least good.	$\leftarrow$	$\mathbf{v}$ is $L(A)$

Figure 4: Ordering and the CRI

Yesterday it was warm	$\rightarrow$	$\mathbf{u}$ is A
Today it is the same.	$\rightarrow$	$\mathbf{u}$ and $\mathbf{v}$ are E
Today it is more or less warm.	$\leftarrow$	$\mathbf{v}$ is $E(A)$

Figure 5: Resemblance and the CRI

Example 6. Suppose that we know that "Fred is a good student" and that "Zdenek is as good or even better." In the universe of students, "good" can be represented by a fuzzy set A, and the relation between the students Zdenek and Fred can be modelled by an ordering L^{-1} representing "is as good or better than." Hence the relation between Fred and Zdenek is given by the inverse L ("is as good or worse than"). According to the CRI, we can then derive the information "Zdenek is $L(A)$." In this paper it is explained that $L(A)$ is a representation for "at least good." In other words, by applying the CRI, we have obtained the result "Zdenek is at least good" without making any computation (i.e. without actually computing the direct image $L(A)$) and without applying any process of linguistic approximation. The derivation process is shown schematically in Figure 4.

Example 7. A similar derivation is shown in Figure 5. In this case, a resemblance relation E is used to model approximate equality in the universe of temperatures. The resulting fuzzy set $E(A)$ can be immediately interpreted as "more or less A".

7 Conclusion

In this paper, we have shown that it is possible to model linguistic modifiers with inherent meaning by taking the mutual relationships between objects of the universe into account. On the formal level, this can be implemented by images of fuzzy sets under fuzzy relations, where three different kinds of images have been studied . In the case of ordering relations and resemblance relations, these images have a clear semantics, and most of them can be used to model linguistic expressions. Furthermore, some of the resulting representations can be smoothly integrated in approximate reasoning schemes using the compositional rule of inference.

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References

[1] R. Babuška. *Fuzzy Modeling for Control.* Kluwer Academic Publishers, Boston/Dordrecht/London, 1998.

[2] W. Bandler and L. Kohout. Fuzzy power sets and fuzzy implication operators. *Fuzzy Sets and Systems*, 4:13–30, 1980.

[3] U. Bodenhofer. The construction of ordering-based modifiers. In G. Brewka, R. Der, S. Gottwald, and A. Schierwagen, editors, *Fuzzy-Neuro Systems '99*, pages 55–62. Leipziger Universitätsverlag, 1999.

[4] U. Bodenhofer. *A Similarity-Based Generalization of Fuzzy Orderings*, volume C 26 of *Schriftenreihe der Johannes-Kepler-Universität Linz.* Universitätsverlag Rudolf Trauner, 1999.

[5] B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order.* Cambridge Mathematical Textbooks. Cambridge University Press, 1990.

[6] M. De Cock and E. E. Kerre. A new class of fuzzy modifiers. In *Proc. ISMVL 2000*, Portland, OR, May 2000. (to appear).

[7] M. De Cock, E. E. Kerre, and A. M. Radzikowska. Modelling linguistic modifiers using fuzzy-rough structures. In *Proc. IPMU 2000*, Madrid, July 2000. (to appear).

[8] S. Gottwald. *Fuzzy Sets and Fuzzy Logic.* Vieweg, Braunschweig, 1993.

[9] H. P. Grice. Further notes on logic and conversation. In P. Cole, editor, *Syntax and Semantics 9: Pragmatics*, pages 113–128. Academic Press, New York, 1978.

[10] E. E. Kerre. *Introduction to the Basic Principles of Fuzzy Set Theory and Some of its Applications.* Communication and Cognition, Gent, 1993.

[11] E. E. Kerre and M. De Cock. Linguistic modifiers: An overview. In G. Chen, M. Ying, and K.-Y. Cai, editors, *Fuzzy Logic and Soft Computing*, pages 69–85. Kluwer Academic Publishers, 1999.

[12] V. Novak and I. Perfilieva. Evaluating linguistic expressions and functional fuzzy theories in fuzzy logic. In L. A. Zadeh and J. Kacprzyk, editors, *Computing with Words in Systems Analysis.* Springer-Verlag, Heidelberg, 1998.

[13] L. A. Zadeh. Calculus of fuzzy restrictions. In L. A. Zadeh, K.-S. Fu, K. Tanaka, and M. Shimura, editors, *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*, pages 1–40. Academic Press, New York, 1975.