REPRESENTATIONS AND CONSTRUCTIONS OF STRONGLY LINEAR FUZZY ORDERINGS

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Summary

This paper is devoted to a class of fuzzy orderings which play a fundamental role in decision analysis and fuzzy control—strongly linear fuzzy (weak) orderings. First, we see that any relation of that kind can be decomposed into a crisp linear ordering and a fuzzy equivalence relation. As a consequence, a general representation theorem follows. Finally, a method for constructing strongly linear fuzzy orderings from pseudo-metrics is presented.

Keywords: Fuzzy ordering, fuzzy weak ordering, preference modeling.

1 INTRODUCTION

Fuzzy (weak) orderings have been studied intensively in the context of preference modeling [3, 5, 6]. Recent investigations [1, 2], moreover, have shown that they can also be integrated fruitfully in areas related to fuzzy systems.

Definition 1. Consider a t-norm T. A binary fuzzy relation $R : X^2 \to [0,1]$ is called

1. *reflexive* iff $\forall x \in X : \; R(x,x) = 1$,
2. *symmetric* iff $\forall x, y \in X : \; R(x,y) = R(y,x)$,
3. *T-transitive* iff
$$\forall x, y, z \in X : \; T\big(R(x,y), R(y,z)\big) \leq R(x,z),$$
4. *strongly linear* (strongly complete) iff
$$\forall x, y \in X : \; \max\big(R(x,y), R(y,x)\big) = 1.$$

Definition 2. Suppose that T is an arbitrary t-norm.

1. A reflexive and T-transitive fuzzy relation is called *fuzzy preordering* with respect to a t-norm T, short *T-preordering.*
2. Symmetric T-preorderings are called *fuzzy equivalence relation* with respect to T, short *T-equivalences.*
3. If a T-preordering is, in addition, strongly linear, it is called *fuzzy weak ordering* with respect to T, for brevity *weak T-ordering.*

It is known, at least if the underlying t-norm T is left-continuous, that T-preorderings and T-equivalences are, in some sense, generated by families of fuzzy subsets of X [8], while an analogous unique representation of fuzzy weak orderings is still missing. The purpose of this paper is (1) to close this gap and (2) to use some of the representation results to link fuzzy weak orderings with pseudo-metrics.

2 REPRESENTATIONS

As a first important result, we can show that any fuzzy weak ordering can be decomposed into a crisp ordering and a fuzzy equivalence relation. In order to be able to do so, we have to introduce the notion of compatibility between a crisp ordering and a fuzzy equivalence relation first.

Definition 3. Let $\lesssim$ be a crisp ordering on X and let E be a fuzzy equivalence relation on X. E is called *compatible with* $\lesssim$, if and only if the following implication holds for all $x, y, z \in X$:
$$x \lesssim y \lesssim z \Longrightarrow E(x,z) \leq \min\big(E(x,y), E(y,z)\big)$$

This property can be interpreted as follows: The two outer elements of a three-element chain are at most as similar as any two inner elements.

Theorem 4. *Consider a fuzzy relation L on a domain X. Then the following two statements are equivalent:*

(i) *L is a weak T-ordering.*

(ii) *There exists a linear ordering $\lesssim$ and a T-equivalence E, which is compatible with $\lesssim$, such that L can be represented as follows:*

$$L(x,y) = \begin{cases} 1 & \text{if } x \lesssim y \\ E(x,y) & \text{otherwise} \end{cases} \tag{1}$$

Proof. See [1]. □

The representation (1) implies that any weak T-ordering is uniquely characterized as a "direct fuzzification" of a crisp linear ordering $\lesssim$, where the fuzzy component can solely be attributed to a T-equivalence.

What follows next is a unique residuum-based characterization of fuzzy weak orderings. For the remaining section, assume that T denotes a left-continuous t-norm with the unique residual implication

$$\vec{T}(x,y) = \sup\{z \in [0,1] \mid T(x,z) \leq y\}$$

and its corresponding biimplication

$$\overleftrightarrow{T}(x,y) = \min\big(\vec{T}(x,y), \vec{T}(y,x)\big).$$

Valverde [8] has shown that any set of fuzzy subsets $(A_i)_{i\in I}$ generates a T-preordering by

$$L(x,y) = \inf_{i\in I} \vec{T}\big(\mu_{A_i}(x), \mu_{A_i}(y)\big). \tag{2}$$

The reverse is true as well, i.e. any T-preordering is generated by some family $(A_i)_{i\in I}$ in the above way.

An analogous representation holds for T-equivalences if we replace the implication $\vec{T}$ by the biimplication $\overleftrightarrow{T}$, i.e. a fuzzy relation E is a T-equivalence if and only if there exists a family of fuzzy subsets $(A_i)_{i\in I}$ such that the following holds [8]:

$$E(x,y) = \inf_{i\in I} \overleftrightarrow{T}\big(\mu_{A_i}(x), \mu_{A_i}(y)\big) \tag{3}$$

For fuzzy weak orderings, however, no such unique representation is available. Several researchers have considered fuzzy relations generated by single fuzzy sets [3, 5]

$$L(x,y) = \vec{T}\big(\mu_A(x), \mu_A(y)\big). \tag{4}$$

As follows from elementary properties of the implication $\vec{T}$, (4) always defines a weak T-ordering, while not every weak T-ordering can be represented in that way.

Theorem 4, however, enables us to formulate a general representation theorem for arbitrary fuzzy weak orderings with respect to left-continuous t-norms.

Theorem 5. *Provided that L is a binary fuzzy relation on a domain X and that T denotes a left-continuous t-norm, the following two statements are equivalent:*

(i) *L is a fuzzy weak ordering with respect to T.*

(ii) *There exists a linear ordering $\lesssim$ on X and a family of fuzzy subsets $(A_i)_{i\in I}$ of X, the membership functions of which are non-decreasing with respect to $\lesssim$, such that the representation (2) holds.*

Proof. *(i)⇒(ii):* First of all, Theorem 4 guarantees the existence of a linear ordering $\lesssim$, which is "below" L, i.e.

$$x \lesssim y \implies L(x,y) = 1. \tag{5}$$

Assuming $I = X$, we define

$$\mu_{x_0}(x) = L(x_0, x)$$

for all $x_0 \in X$. Since L is T-transitive and property (5) holds, $x \lesssim y$ implies

$$L(x_0,x) = T\big(L(x_0,x), \overbrace{L(x,y)}^{=1}\big) \leq L(x_0,y)$$

which equivalent to $\mu_{x_0}(x) \leq \mu_{x_0}(y)$ and we have shown that all membership functions μ_{x_0} are non-decreasing w.r.t. $\lesssim$. Then the representation (2) follows from reflexivity and T-transitivity as in the proof of [8, Theorem 4.1].

(ii)⇒(i): Analogous to the proof of [8, Theorem 4.1], where strong linearity immediately follows from the monotonicity of all μ_{A_i}. □

It is worth to mention that, if we start from a given weak T-ordering L, neither the ordering $\lesssim$ nor the family $(A_i)_{i\in I}$ need to be unique.

Finally, we can adapt the representation (3) to the case of compatibility with a given crisp linear ordering.

Theorem 6. *Consider a crisp linear ordering $\lesssim$ of the domain X and a fuzzy relation $E : X^2 \to [0,1]$. Then the following two statements are equivalent:*

(i) *E is a T-equivalence which is compatible with $\lesssim$.*

(ii) *There exists a family of fuzzy subsets $(A_i)_{i\in I}$, the membership functions of which are non-decreasing with respect to $\lesssim$, such that the representation (3) holds.*

Proof. *(i)⇒(ii):* First of all, by (1), we can define a weak T-ordering L. It is easy to see that E is exactly the symmetric kernel of L:

$$E(x,y) = \min\big(L(x,y), L(y,x)\big)$$

Since, for the relation L, Theorem 5 is applicable, we obtain

$$\begin{aligned} E(x,y) &\min\big(L(x,y), L(y,x)\big) \\ &= \min\big(\inf_{i\in I} \vec{T}(\mu_{A_i}(x), \mu_{A_i}(y)), \\ &\qquad \inf_{i\in I} \vec{T}(\mu_{A_i}(y), \mu_{A_i}(x))\big) \\ &= \inf_{i\in I} \min\big(\vec{T}(\mu_{A_i}(x), \mu_{A_i}(y)), \\ &\qquad \vec{T}(\mu_{A_i}(y), \mu_{A_i}(x))\big) \\ &= \inf_{i\in I} \overleftrightarrow{T}\big(\mu_{A_i}(x), \mu_{A_i}(y)\big). \end{aligned}$$

(ii)⇒(i): Similar to [8, Theorem 4.2], where the compatibility easily follows from the monotonicity of all μ_{A_i}. □

3 CONSTRUCTIONS

In this section, we will discuss in which way strongly linear fuzzy orderings can be constructed provided that some linear ordering of the given universe is known. Of course, Theorem 5 provides the most flexible way to construct such orderings. Alternatively, we will now consider how, by means of Theorem 4, strongly linear fuzzy orderings can be constructed from pseudo-metrics, at least if the considered t-norm T is continuous and Archimedean, i.e.

$$\forall x \in (0,1) : \ T(x,x) < x.$$

We will utilize the well-known correspondence between pseudo-metrics and T-equivalences (e.g. [4]) and extend this concept in order to integrate compatibility with a given crisp linear ordering.

Before turning to the core, let us briefly recall some important preliminaries.

Theorem 7. *A function $T : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-norm if and only if there exists a continuous, strictly decreasing function $f : [0,1] \to [0,\infty]$ with $f(1) = 0$ called* additive generator*such that, for all $x, y \in [0,1]$, the following holds:*

$$T(x,y) = f^{-1}\big(\min(f(x) + f(y), f(0))\big)$$

The generator f is uniquely determined up to a positive multiplicative constant.

Proof. See e.g. [7]. □

Definition 8. A mapping $d : X^2 \to [0,\infty]$ is called *pseudo-metric* on X if and only if the following axioms hold:

1. *Homogeneity:* $\forall x \in X : \ d(x,x) = 0$
2. *Symmetry:* $\forall x, y \in X : \ d(x,y) = d(y,x)$
3. *Triangle inequality:*

$$\forall x, y, z \in X : \ d(x,z) \leq d(x,y) + d(y,z)$$

Theorem 9. *Assume that T is an Archimedean t-norm with an additive generator f.*

1. *For any pseudo-metric $d : X^2 \to [0,\infty]$, the mapping*

$$E_d(x,y) = f^{-1}\big(\min(d(x,y), f(0))\big) \qquad (6)$$

defines a T-equivalence on X.

2. *Provided that E is a T-equivalence on X, we can define a pseudo-metric d_E as*

$$d_E(x,y) = f(E(x,y)). \qquad (7)$$

Proof. See [4]. □

The following lemma induces a dual notion of compatibility between pseudo-metrics and orderings and characterizes its correspondence with the compatibility between fuzzy equivalence relations and orderings.

Lemma 10. *Let T be a continuous Archimedean t-norm with an additive generator f and let $\lesssim$ be an ordering of the domain X.*

1. *If a pseudo-metric d on X has the property*

$$\forall x, y, z \in X : \ x \lesssim y \lesssim z \Longrightarrow \\ d(x,z) \geq \max(d(x,y), d(y,z)), \qquad (8)$$

then its induced fuzzy equivalence relation E_d, defined as in (6), is compatible with $\lesssim$.

2. *If a fuzzy equivalence relation E is compatible with $\lesssim$, its induced pseudo-metric d_E, defined as in (7), fulfills property (8).*

Proof. Follows directly from the fact that the additive generator f and its inverse are non-increasing function. □

Moreover, we obtain that all desired properties remain even after transformation with some monotonic (not necessarily bijective) function.

Lemma 11. *Consider a pseudo-metric $d : X^2 \to [0,1]$, which is compatible with a linear ordering $\lesssim$, and a non-decreasing function $\varphi : X \to X$. Then the mapping*

$$d'(x,y) = d(\varphi(x), \varphi(y))$$

also defines a pseudo-metric on X which is compatible with $\lesssim$.

Proof. Trivial. □

Putting all the above results together, we obtain the following construction theorem.

Theorem 12. *Assume that we are given a crisp linear ordering $\lesssim$ of a space X, a pseudo-metric d, which is compatible with $\lesssim$ in the sense of (8), a continuous Archimedean t-norm T with an additive generator f, and a non-decreasing function $\varphi : X \to X$. Then*

$$L_{\varphi,d}(x,y) = \begin{cases} 1 & \text{if } x \lesssim y \\ E_{\varphi,d}(x,y) & \text{otherwise} \end{cases}$$

defines a strongly linear fuzzy ordering with respect to T and

$$E_{\varphi,d}(x,y) = f^{-1}\big(\min(d(\varphi(x),\varphi(y)), f(0))\big).$$

Proof. Immediate consequence of Theorem 4 and the above results. □

Example 13. It is easy to see that $d(x,y) = |x-y|$ is a (pseudo-)metric on the real numbers which is compatible with their natural ordering $\leq$. It is, moreover, straightforward to check that $1-x$ is a self-inverse additive generator of the Łukasiewicz t-norm

$$T_{\mathbf{L}}(x,y) = \max(x+y-1, 0)$$

and that e^{-x} is an additive generator of the product t-norm $T_{\mathbf{P}}$, whose inverse is $-\ln x$. Hence, for any non-decreasing function $\varphi : \mathbb{R} \to \mathbb{R}$, we obtain that

$$\begin{aligned} E'_{\varphi}(x,y) &= 1 - (\min(|\varphi(x) - \varphi(y)|, 1)) \\ &= \max(1 - |\varphi(x) - \varphi(y)|, 0) \end{aligned}$$

is a $T_{\mathbf{L}}$-equivalence which is compatible with $\leq$ and that

$$E''_{\varphi}(x,y) = e^{-\min(|\varphi(x)-\varphi(y)|,\infty)} = e^{-|\varphi(x)-\varphi(y)|}$$

is a $T_{\mathbf{P}}$-equivalence which is compatible with $\leq$. Then, based on Theorem 12, we can define strongly linear fuzzy orderings on the real numbers, which fuzzify the linear ordering $\leq$.

Not surprisingly, we can also revert this construction: Starting from an arbitrary weak T-ordering L, by means of Theorem 4, we obtain a linear ordering $\lesssim$ and a T-equivalence E compatible with $\lesssim$. Then, by Theorem 9 and Lemma 10, we can construct a pseudo-metric which is compatible with $\lesssim$ as well.

4 CONCLUDING REMARKS

We have seen in which way fuzzy weak orderings are characterized. On the one hand, they can be considered as combinations (unions) of crisp linear orderings and fuzzy equivalence relations. On the other hand, they are uniquely characterized as (possibly infinite) min-intersections of fuzzy weak orderings induced by fuzzy sets with monotonic membership functions. Perhaps the most important discovery is that, even if intermediate degrees of preference are allowed, there is an underlying crisp ordering.

In the second part—soundly extending the well-known connection between pseudo-metrics and fuzzy equivalence relations—we have considered the correspondence between fuzzy weak orderings and pseudo-metrics.

The next step is to study the consequences of these results in decision making, in particular concerning aggregation of preferences.

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