Adding Similarity to Fuzzy Orderings

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Abstract

The objective of this contribution is to point out problems in existing definitions of fuzzy orderings and, consequently, to present an alternative similarity-based approach.

Keywords: fuzzy ordering; similarity relation.

1 Introduction

Needless to say, higher mathematics would be unthinkable without orderings and equivalence relations. They, however, play a fundamental role even when it concerns fuzzy systems which were introduced with the objective to model human-like decisions by taking the graduality of human thinking and reasoning into account.

One may easily observe that gradual similarity is an inherent component of fuzziness. It is nowadays a well-accepted fact that fuzzy equivalence relations are appropriate models of gradual similarity [6, 10, 12].

Definition 1. A binary fuzzy relation E on a domain X is called *fuzzy equivalence relation* with respect to a t-norm T, for brevity T*-equivalence*, if and only if the following three axioms are fulfilled for all $x, y, z \in X$:

1. Reflexivity: $E(x,x) = 1$
2. Symmetry: $E(x,y) = E(y,x)$
3. T-transitivity: $T(E(x,y), E(y,z)) \leq E(x,z)$

Fuzzy equivalence relations have turned out to be helpful tools in various disciplines, in particular, when it concerns the interpretation of fuzzy sets, partitions, and controllers [6, 7, 10].

Since graduality also appears in the way humans specify preferences, it would be useful to have a model of gradual ordering, too. Possible applications could be found in fuzzy decision analysis, but also in areas which are considered to be related to fuzzy systems and fuzzy control, such as rule interpolation and linguistic approximation.

It is near at hand to define fuzzy orderings by taking appropriate fuzzifications of the three classical axioms as in the case of equivalence relations. The first definition of that type already appears in [12] under the name *fuzzy partial ordering*.

Definition 2. A reflexive and T-transitive fuzzy relation L is called *fuzzy ordering* with respect to a t-norm T, short T*-ordering*, if and only if it additionally fulfills T-antisymmetry, i.e., for all $x, y \in X$,

$$x \neq y \Rightarrow T(L(x,y), L(y,x)) = 0.$$

L is called *strongly linear* if and only if, for all $x, y \in X$,

$$\max(L(x,y), L(y,x)) = 1.$$

A reflexive and T-transitive fuzzy relation is called *fuzzy preordering* with respect to T, for brevity T*-preordering*.

In the following, by the help of three case studies, we will try to give reasons why the above concept of gradual ordering appears to be counter-intuitive and, therefore, inappropriate for real-world applications. Based on these considerations, a generalization will be proposed which also takes the strong connection between similarity and ordering into account.

2 A Critical View

2.1 Implications as Orderings?

It is a well-known and often-used fact in mathematical logic that there is a strong connection between implications and orderings. Consider, for instance, the relation

$$\varphi \lesssim \psi \iff (\varphi \to \psi \text{ is a tautology}),$$

where φ and ψ are formulas. It is easy to see that $\lesssim$ defines an ordering of the set of formulas if we always consider two formulas as equal if their evaluations coincide for all interpretations.

The same is true for many-valued propositional logics based on residuated lattices [5] which also comprise t-norm-based logics on the unit interval. For details and proofs, the reader is referred to [3, 4, 5].

Definition 3. Consider an arbitrary t-norm T. A mapping $R : [0,1]^2 \to [0,1]$ is called *residual implication (residuum)* of T if and only if the following equivalence is fulfilled for all $x, y, z \in [0,1]$:

$$T(x,y) \leq z \iff x \leq R(y,z)$$

Lemma 4. *For any left-continuous t-norm T, there exists a unique residuum $\vec{T}$ given as*

$$\vec{T}(x,y) = \sup\{u \in [0,1] \mid T(u,x) \leq y\}.$$

Lemma 5. *If T is a left-continuous t-norm, the following holds for all $x, y, z \in [0,1]$:*

1. $x \leq y \iff \vec{T}(x,y) = 1$
2. $T\big(\vec{T}(x,y), \vec{T}(y,z)\big) \leq \vec{T}(x,z)$
3. $\vec{T}(1,y) = y$

If we consider $\vec{T}$ as a binary relation on the unit interval, the equivalence 1. above states that the kernel relation of $\vec{T}$ coincides with the crisp ordering $\leq$. Now the question arises whether $\vec{T}$ is even a fuzzy ordering in the sense of Definition 2. The answer is simple: Lemma 5 yields that $\vec{T}$ is reflexive and T-transitive. On the other hand, T-antisymmetry would imply

$$T\big(\vec{T}(x,y), \vec{T}(y,x)\big) = 0$$

for any $x \neq y$, which can never be fulfilled due to 3. in Lemma 5. Hence, we obtain that there is no left-continuous t-norm such that its residuum is a fuzzy ordering.

2.2 Inclusion Relations

It should be known that, for each non-empty crisp set X, the inclusion $\subseteq$ is an ordering of the power set $\mathcal{P}(X)$. The same holds in the fuzzy case, i.e. the well-known inclusion

$$A \subseteq B \iff \big(\forall x \in X : \ \mu_A(x) \leq \mu_B(x)\big)$$

defines an ordering of the fuzzy power set $\mathcal{F}(X)$.

Let us try to define a fuzzy concept of inclusion and check whether it can be a fuzzy ordering. In the crisp case, we can rewrite $A \subseteq B$ as

$$\forall x \in X : \ x \in A \Longrightarrow x \in B.$$

Fixing a certain left-continuous t-norm T, we can interpret this formula in the setting of fuzzy predicate logic [5] even if A and B are fuzzy subsets of X. Then the degree of inclusion can be computed as follows [1, 4]:

$$\mathrm{INCL}_T(A,B) = \inf_{x \in X} \vec{T}(\mu_A(x), \mu_B(x)).$$

Lemma 6. *For an arbitrary left-continuous t-norm T, the fuzzy relation INCL_T is a T-preordering on $\mathcal{F}(X)$.*

Proof. Follows from Lemma 5 and elementary properties of t-norms. □

Concerning T-antisymmetry, the answer is again negative. To see that, take an arbitrary element $\tilde{x} \in X$ and define $A = \{\tilde{x}\}$ and B as

$$\mu_B(x) = \begin{cases} r & \text{if } x = \tilde{x}, \\ 0 & \text{otherwise,} \end{cases}$$

with a fixed $r \in (0,1)$. Then the following degrees of inclusion are obtained:

$$\mathrm{INCL}_T(A,B) = \vec{T}(1,r) = r$$
$$\mathrm{INCL}_T(B,A) = 1$$

Since $T(1,r) = r > 0$, we have shown that there is no left-continuous t-norm T such that INCL_T is T-antisymmetric. Therefore, INCL_T cannot be a fuzzy ordering, regardless of the t-norm chosen.

2.3 The Fuzzification Property

The purpose of this subsection is to introduce a criterion for checking whether a fuzzy relation is a "fuzzification" of a crisp relation. Subsequently, we will apply this criterion to fuzzy equivalence relations and fuzzy orderings.

Definition 7. Consider a crisp binary relation $\diamondsuit$ on a set X. A fuzzy relation R is called $\diamondsuit$-*consistent* if and only if the implication

$$y \diamondsuit z \Longrightarrow R(x,y) \leq R(x,z)$$

holds for all $x, y, z \in X$. If, additionally, $\chi_{\diamondsuit} \leq R$ holds, equivalently,

$$x \diamondsuit y \Longrightarrow R(x,y) = 1,$$

R is called a *fuzzification* of $\diamondsuit$ ("R *fuzzifies* $\diamondsuit$").

Now let us examine the two prominent classes of fuzzy relations.

Proposition 8. *Every T-equivalence E fuzzifies its kernel equivalence relation defined as*

$$x \sim_E y \iff E(x,y) = 1.$$

Proof. See [2]. □

Before considering fuzzifications of orderings, let us take a closer look at the consistency property in the case of a crisp ordering $\lesssim$:

$$y \lesssim z \Longrightarrow R(x,y) \leq R(x,z) \qquad (1)$$

This means that a fuzzy relation R is $\lesssim$-consistent if and only if each vertical cut $R(x,.)$ is non-decreasing, i.e. the degree to which a value y is "smaller or equal" to x is non-decreasing—a property one would naturally demand of a fuzzification of $\lesssim$. The next result, however, shows that there are no non-trivial fuzzifications of crisp linear orderings.

Proposition 9. *Let $\lesssim$ be a crisp linear ordering of X. Then $\lesssim$ itself is the only fuzzy ordering which is $\lesssim$-consistent and, as a consequence, the only fuzzification of $\lesssim$.*

Proof. Assume that a fuzzy ordering L is $\lesssim$-consistent. Taking reflexivity and consistency (1) into account, we obtain

$$\forall y \gtrsim x : \; L(x,y) = 1.$$

Then T-antisymmetry and linearity imply

$$\forall y \not\gtrsim x : \; L(x,y) = 0$$

and we have shown that $L = \chi_{\lesssim}$. □

This result drastically shows that fuzzy orderings in the sense of Definition 2 are not even able to provide consistent fuzzifications of the linear ordering of real numbers.

2.4 Reflexivity versus Antisymmetry

It is easy to see from the proof of Proposition 9, that the problems in terms of consistency can be avoided if either reflexivity or antisymmetry is dropped. Although this is never mentioned explicitly, one may suspect that the researchers have had that problems in mind when they proposed, for instance, non-reflexive fuzzy orderings [3, 9, 12]. They went the easier way: Antisymmetry is considered as *the* fundamental property of orderings. Even if reflexivity is omitted, an ordering-like structure (consider, e.g. strict orderings) can be obtained—hence, the conflict between reflexivity and antisymmetry was solved by the omission of reflexivity. On the contrary, the difficulties concerning implications and inclusions are not resolved by dropping reflexivity.

The author is deeply convinced that simply omitting axioms, however, does not solve the problem sufficiently, since the axioms of orderings proved to be appropriate for a long time; every single one has its own justification—omitting just opens the field for arbitrariness. Moreover, it seems to be an eyesore that many properties carry over to the fuzzy variant in the case of equivalence relations, but not in the case of orderings.

Assuming that an approach is desirable, which includes all three classical axioms, but solves all the above problems, let us try to find the actual reasons for the difficulties. The definitions of reflexivity and T-transitivity are, more or less, straightforward. So, we should take a closer look at T-antisymmetry, which is obviously equivalent to

$$T(L(x,y), L(y,x)) \leq \chi_{=}(x,y). \tag{2}$$

One immediately sees that this, indeed, seems to be an appropriate fuzzification of the classical axiom of antisymmetry

$$(x \leq y \wedge y \leq x) \Longrightarrow x = y, \tag{3}$$

where the crisp ordering $\leq$ is replaced by the fuzzy ordering L. The crisp equality on the right-hand side, however, remains untouched. This actually means that, even if two values are almost indistinguishable, they have to be ranked, more or less, crisply. In this sense, the definition of T-antisymmetry is a kind of "half-way fuzzification".

Observation 10. *Evidently, the above example shows that requiring crisp equality in the definition of T-antisymmetry seems to contradict to the nature of vague environments. This is even less surprising if we think of orderings as mathematical models of expressions, such as "smaller/greater or equal", and, consequently, of fuzzy orderings as models of vague expressions, such as "approximately smaller/greater or equal", where one immediately sees the inherent component of similarity. This entails the requirement on fuzzy orderings to take gradual similarity into account.*

The most obvious way to overcome all the problems seems to replace the crisp equality in (2) by a fuzzy concept of equality—a fuzzy equivalence relation. As a consequence, if a fuzzy ordering should respect similarity, the distinction between two values should not be stricter than that provided by the fuzzy equivalence relation. Actually, this means that, following the equivalent formulation of crisp reflexivity

$$\forall x, y \in X : \; x = y \Longrightarrow x \leq y,$$

the crisp equality should also be replaced by a fuzzy equivalence relation.

3 Preserving the Classical Axioms by Adding Similarity

According to the discussions in the previous section, we can finally define the similarity-based generalization.

Definition 11. Let $L : X^2 \to [0,1]$ be a T-transitive fuzzy relation. L is called *fuzzy ordering* with respect to a t-norm T and a T-equivalence E, for brevity *T-E-ordering*, if and only if it additionally fulfills the following two axioms for all $x, y \in X$:

1. E-reflexivity: $\quad E(x,y) \leq L(x,y)$
2. T-E-antisymmetry:

$$T(L(x,y), L(y,x)) \leq E(x,y)$$

Before turning to more sophisticated considerations, let us briefly check in which way the above modification relates to the existing concepts of crisp and fuzzy orderings. The following equivalence holds trivially:

$$L(x,x) = 1 \iff \big(\forall y \in X : \; \chi_{=}(x,y) \leq L(x,y)\big)$$

Hence, reflexivity is equivalent to $\chi_{=}$-reflexivity, while inequality (2) states that T-antisymmetry is equivalent to T-$\chi_{=}$-antisymmetry. We obtain that every T-ordering in the sense of Definition 2 fulfills the axioms of Definition 11 with $E = \chi_{=}$. From this point of view, the new definition of T-E-orderings generalizes the existing concept of T-orderings just by admitting an additional degree of freedom—the fuzzy equivalence relation E. Moreover, one easily sees that all crisp orderings are T-$\chi_{=}$-orderings, where T can be chosen arbitrarily.

It is easy to prove that the symmetric kernel of a crisp preordering is an equivalence relation. A fuzzy analogue has been proved in [10]. The following result shows, one step further, how to construct T-equivalences such that a given T-preordering can be considered as a fuzzy ordering.

Theorem 12. *A T-preordering L is a fuzzy ordering with respect to T and a T-equivalence E if and only if, for all $x, y \in X$,*

$$\begin{aligned} T(L(x,y), L(y,x)) &\leq E(x,y) \\ &\leq \min(L(x,y), L(y,x)). \end{aligned}$$

The two bounds are T-equivalences themselves.

Proof. Obviously, the lower bound directly corresponds to T-E-antisymmetry while the upper bound is equivalent to E-reflexivity. The last assertion follows from elementary properties of t-norms [10]. $\square$

Adopting this point of view naively, the new approach seems to result in the hidden removal of the antisymmetry axiom. Yet this is only true if one does not care about the choice of the underlying fuzzy equivalence relation E. If, however, a certain notion of similarity in a certain vague environment is assumed in advance, T-E-antisymmetry has a concrete meaning—that the degree of "non-antisymmetry" is bounded by the degree of similarity.

The existence of an E such that a T-preordering L is a T-E-ordering only implies that L can be considered as a reasonable concept of ordering if one can consider E as a reasonable concept of similarity in the given environment—otherwise the relation E is of no practical use and its introduction is purely artificial. In this sense, the above theorem provides a criterion for checking whether a given fuzzy preordering has a reasonable interpretation as fuzzy ordering.

In any case, we should not neglect that the same problems can appear even in the crisp case. Considering the example of formulas given in 2.1, it is easy to recognize that it is definitely not always a trivial task to specify a proper concept of equality. In many cases, the term "equal" is nonchalantly used when meaning "equivalent".

Example 13. In 2.1 and 2.2, two binary fuzzy relations were discussed, which could be regarded as fuzzy orderings intuitively, but turned out to violate T-antisymmetry. Now, in the more general framework, Theorem 12 guarantees that there are fuzzy equivalence relations such that both can be interpreted as fuzzy orderings. According to the above discussions, we have to check whether the induced fuzzy equivalence relations are reasonable concepts of similarity.

First of all, for an arbitrary left-continuous t-norm T, we obtain that $\vec{T}$ is indeed a fuzzy ordering with respect to T and

$$\overset{\leftrightarrow}{T}(x,y) = T(\vec{T}(x,y), \vec{T}(y,x))$$

which is well-known as the so-called biimplication corresponding to T. Since $\overset{\leftrightarrow}{T}$ is almost the only imaginable concept of equivalence in logical terms, we see that $\vec{T}$ can be interpreted seriously as a fuzzy ordering which is even strongly linear. Therefore, the problems stated in 2.1 are perfectly solved in the new framework.

Now let us consider the symmetric kernel of the inclusion relation INCL_T, where, according to Theorem 12, we take the upper bound:

$$\begin{aligned} &\min\left(\inf_{x\in X} \vec{T}(\mu_A(x), \mu_B(x)), \inf_{x\in X} \vec{T}(\mu_B(x), \mu_A(x))\right) \\ &= \inf_{x\in X} \min\left(\vec{T}(\mu_A(x), \mu_B(x)), \vec{T}(\mu_B(x), \mu_A(x))\right) \\ &= \inf_{x\in X} \overset{\leftrightarrow}{T}(\mu_A(x), \mu_B(x)) \end{aligned}$$

This fuzzy equivalence relation is well-known for measuring the similarity of fuzzy sets, at least for the Łukasiewicz t-norm [8, 11]. So, we have resolved all the difficulties concerning inclusion relations, too.

It remains to investigate whether the new class of fuzzy orderings is able to provide consistent fuzzifications of crisp (linear) orderings without diminishing to trivial cases as in 2.3.

Lemma 14. *For any T-E-ordering L, the kernel relation*

$$x \trianglelefteq_L y \Longleftrightarrow L(x,y) = 1$$

defines a preordering. L fuzzifies $\trianglelefteq_L$ and all crisp orderings it contains.

Proof. Trivially, $\trianglelefteq_L$ is a preordering. Consistency follows from the definition of $\trianglelefteq_L$ and T-transitivity, while the last assertion follows directly from the definition of consistency. $\square$

According to the problems stated in 2.3, we still have to check in which way fuzzifications of crisp linear orderings can be obtained.

Definition 15. Let $\lesssim$ be a crisp ordering on X and let E be a fuzzy equivalence relation on X. E is called *compatible with* $\lesssim$, if and only if the following implication holds for all $x, y, z \in X$:

$$x \lesssim y \lesssim z \Longrightarrow (E(x,z) \leq \min\big(E(x,y), E(y,z)\big).$$

This property can be interpreted as follows: The two outer elements of a three-element chain are at most as similar as any two inner elements.

Theorem 16. *Consider a fuzzy relation L on a domain X and a T-equivalence E. Then the following two statements are equivalent:*

1. *L is a strongly linear T-E-ordering.*
2. *There exists a linear ordering $\lesssim$ the relation E is compatible with such that L can be represented as follows:*

$$L(x,y) = \begin{cases} 1 & \textit{if } x \lesssim y \\ E(x,y) & \textit{otherwise} \end{cases}$$

Proof. See [2]. □

Theorem 16 states that strongly linear fuzzy orderings are uniquely characterized as fuzzifications of crisp linear orderings, where the fuzzy component can be attributed to a fuzzy equivalence relation, and we have also solved the problems concerning fuzzifications.

4 Concluding Remarks

The aim of this paper was to motivate the need for a new approach to fuzzy orderings which also integrates similarity. By the help of three case studies, we have seen that properties one would naturally demand of a gradual concept of ordering are violated for the existing approach but can be fulfilled for the similarity-based generalization.

In [2], these considerations are the starting point for the investigation of the following topics:

1. Constructions and representations of the new class of fuzzy orderings.
2. Definition and detailed investigation of hull operators which can be used to define ordering-based modifiers, such as *'at least'*, *'at most'*, *'between'*, and so forth.
3. As an important application, hulls with respect to fuzzy orderings provide a way to define orderings of fuzzy sets even if no assumptions concerning linearity are made with the additional advantage that similarity can be taken into account.

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