The Construction of Ordering-Based Modifiers

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Abstract

Ordering-based modifiers can be particularly useful in fuzzy control and other fields related to fuzzy systems. This paper deals with the construction of such modifiers by means of hull operators with respect to fuzzy orderings.

Key words: *fuzzy orderings, hull operators, modifiers.*

1 Introduction

Already in their beginning, fuzzy systems were considered as appropriate tools for controlling complex systems and for carrying out complicated decision processes [12]. It is well-known and easy to see that, if rule bases are represented as complete tables, the number of rules grows exponentially with the number of variables—a fact which can be regarded as a serious limitation in terms of surveyability and interpretability.

Almost all fuzzy systems make implicit use of orderings. More specifically, it is quite common to decompose the universe of a system variable into a certain number of fuzzy subsets by means of the ordering of the universe—an approach which is often reflected in labels like *`small'*, *`medium'*, or *`large'*. We will now demonstrate by means of a simple example how such information can be used to reduce the size of a rule base while improving expressiveness and interpretability. Consider a typical PD-style fuzzy controller with two inputs e, $\dot{e}$ and one output variable f, where the universes of all these variables are covered by five fuzzy sets labeled *`NB'*, *`NS'*, *`Z'*, *`PS'*, and *`PB'*:

$e \backslash \dot{e}$	*NB*	*NS*	*Z*	*PS*	*PB*
NB	*NB*	*NB*	*NB*	*NS*	*Z*
NS	*NB*	*NB*	*NS*	*Z*	*PS*
Z	*NB*	*NS*	*Z*	*PS*	*PB*
PS	*NS*	*Z*	*PS*	*PB*	*PB*
PB	*Z*	*PS*	*PB*	*PB*	*PB*

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One possibility to reduce the size of this rule base is to take neighboring rules with the same consequents, such as,

$$\begin{array}{llll} \text{IF} \quad e \text{ is } `NB' & \text{AND} \quad \dot{e} \text{ is } `NB' & \text{THEN} \quad f \text{ is } `NB' \\ \text{IF} \quad e \text{ is } `NS' & \text{AND} \quad \dot{e} \text{ is } `NB' & \text{THEN} \quad f \text{ is } `NB' \\ \text{IF} \quad e \text{ is } `Z' & \text{AND} \quad \dot{e} \text{ is } `NB' & \text{THEN} \quad f \text{ is } `NB' \end{array}$$

and to replace them by a single rule like the following one:

$$\text{IF} \quad e \text{ is } `\textit{at most } Z' \quad \text{AND} \quad \dot{e} \text{ is } `NB' \quad \text{THEN} \quad f \text{ is } `NB'$$

Of course, there is actually no need to do so in such a simple case. Anyway, grouping neighboring rules by means of expressions, such as, *`at least'*, *`at most'*, or *`between'*, could help to reduce the size of larger rule bases considerably.

In addition, such elements can be useful in rule interpolation. Sometimes, when experts or automatic tuning procedures only provide an incomplete description of a fuzzy rule base, it can still be necessary to obtain a conclusion even if an observation does not match any antecedent in the rule base [5]. Moreover, it is considered as another opportunity for reducing the size of a rule base to store only some representative rules and to interpolate between them [6]. In any case, it is indispensable to have criteria for determining between which rules the interpolation should take place. Beside distance, orderings play a fundamental role in this selection. As an alternative to distance-based methods [6], it is possible to fill the gap between the antecedents of two rules using a fuzzy concept of *`between'*, which leads us to the ordering-based modifiers mentioned above.

The fact remains that we are still lacking a way how to compute such expressions in the presence of fuzziness. In order to have a universal approach, which is applicable in a wide variety of practical problems, at least the following two properties should be satisfied:

(i) If there is a predefined notion of fuzzy similarity in the given environment, the above operators should take it into account. Considering the example of the height of people, this means that *`at least* 180' should not exclude 179.9 completely, because the two values are almost indistinguishable.

(ii) For using these expressions as modifiers in the language of a rule-based fuzzy system, they should be applicable to fuzzy sets, because the atomic expressions are usually represented by fuzzy sets instead of crisp values.

Let us consider the case of a crisp ordering $\preceq$ first. For a single crisp value x, the set *`at least* $x$' can be defined as $\{y \mid x \preceq y\}$. This notion can be generalized to crisp sets, where, for an arbitrary crisp set $M$, *`at least* M' can be defined as follows:

$$\{y \mid \exists x \in M : \; x \preceq y\} \tag{1}$$

We will discuss a way how to generalize the definition in Equation (1) such that, firstly, it can be applied to fuzzy sets and, secondly, the crisp ordering $\preceq$ can be replaced by a fuzzy concept of ordering which also takes indistinguishability into account.

In the following, assume that all t-norms we deal with are at least left-continuous and that the symbols $\cap$, $\cup$, and $\complement$, as usual, denote minimum intersection, maximum union, and the fuzzy complement with respect to the standard negation $N_{\mathbf{S}}(x) = 1 - x$, respectively.

2 Fuzzy Orderings and their Hull Operators

We have mentioned already that it can be desirable to consider gradual similarity, too. Hence, before going into more detail, we briefly recall *the* common concept of fuzzy similarity or indistinguishability—fuzzy equivalence relations [4, 9, 11].

Definition 1. A mapping $E : X^2 \to [0,1]$ is called a *fuzzy equivalence relation* on the domain X with respect to a t-norm T, short *T-equivalence*, if and only if it satisfies the following axioms:

$$\begin{array}{lll} \forall x \in X : & E(x,x) = 1 & \text{(reflexivity)} \\ \forall x,y \in X : & E(x,y) = E(y,x) & \text{(symmetry)} \\ \forall x,y,z \in X : & T(E(x,y),E(y,z)) \leq E(x,z) & (T\text{-transitivity}) \end{array}$$

Now we turn to the definition of a gradual concept of ordering which also takes the strong connection between similarity and ordering into account [1].

Definition 2. A function $L : X^2 \to [0,1]$ is called a *fuzzy ordering* on X with respect to a t-norm T and a T-equivalence E, for brevity *T-E-ordering*, if and only if it satisfies the following three axioms:

$$\begin{array}{lll} \forall x,y \in X : & E(x,y) \leq L(x,y) & (E\text{-reflexivity}) \\ \forall x,y \in X : & T(L(x,y),L(y,x)) \leq E(x,y) & (T\text{-}E\text{-antisymmetry}) \\ \forall x,y,z \in X : & T(L(x,y),L(y,z)) \leq L(x,z) & (T\text{-transitivity}) \end{array}$$

L is called *strongly linear* if and only if , for every pair (x,y), either $L(x,y) = 1$ or $L(y,x) = 1$ holds.

For more details on fuzzy orderings, their properties and applications, the reader is referred to [1]. We just mention that, by replacing the fuzzy equivalence relation E by the crisp equality, the well-known definition of fuzzy partial orderings [11] is obtained. Moreover, one easily verifies that this still includes crisp orderings as well.

After providing these basics, we can now turn back to the problem of generalizing (1). Rewriting this definition a little, we obtain

$$x \in \text{`}\textit{at least } M\text{'} \iff \big(\exists y \in X : \, y \in M \wedge y \preceq x\big).$$

If we replace M by a fuzzy set A and $\preceq$ by a fuzzy ordering L, it just remains to define proper fuzzifications of the existential quantifier and the Boolean conjunction $\wedge$. Closely related to fuzzy predicate logic [3], we will take the supremum and the underlying t-norm T, respectively, which results in the following generalization:

$$\mu_{\text{`}\textit{at least } A\text{'}}(x) = \sup\{T(\mu_A(y), L(y,x)) \mid y \in X\} \tag{2}$$

Actually, this is nothing else than the hull of A with respect to L [7], alternatively called (full) image of A under L [2]. For simplicity, we will abbreviate this operator with ATL.

Analogously, it is possible to define a fuzzy concept of *`at most'* just by taking the inverse ordering $G(x,y) = L(y,x)$

$$\mu_{\text{`}\textit{at most } A\text{'}}(x) = \sup\{T(\mu_A(y), L(x,y)) \mid y \in X\} \tag{3}$$

which will be denoted ATM in the following.

3 Basic Properties of Ordering-Based Modifiers

The question arises whether there is a correspondence between the modifiers ATL and ATM and the so-called extensional hull—the hull with respect to the underlying fuzzy equivalence relation E. We will show that, if a fuzzy ordering can be represented as the union of a crisp ordering and a fuzzy equivalence relation, this representation carries over to the operators ATL and ATM as well. For proof details, the reader is referred to [1].

In the following, assume that L is a given T-E-ordering, where the t-norm T and the fuzzy equivalence relation E are supposed to be fixed.

Definition 3. A fuzzy set A is called *extensional* (with respect to E) if and only if

$$\forall x,y \in X: \ T(\mu_A(x), E(x,y)) \leq \mu_A(y).$$

The smallest extensional superset of A is called *extensional hull* of A and denoted with $\mathrm{EXT}(A)$.

It is rather easy to show [7] that—in analogy to (2) and (3)—the following representation holds:

$$\mu_{\mathrm{EXT}(A)}(x) = \sup\{T(\mu_A(y), E(y,x)) \mid y \in X\}$$

Definition 4. Provided that the domain X is equipped with some crisp ordering $\preceq$ (not necessarily linear), a fuzzy subset A of X is called *convex* (compare with [8, 10]) if and only if

$$\forall x,y,z \in X: \ x \preceq y \preceq z \Longrightarrow \mu_A(y) \geq \min(\mu_A(x), \mu_A(z)).$$

Proposition 5. *For any fuzzy set A, the sets* $\mathrm{ATL}(A)$ *and* $\mathrm{ATM}(A)$ *are extensional. Moreover, if a crisp ordering $\preceq$ is a subrelation of L, i.e.,*

$$\forall x,y \in X: \ x \preceq y \Longrightarrow L(x,y) = 1,$$

then $\mathrm{ATL}(A)$ *and* $\mathrm{ATM}(A)$ *are convex.*

Definition 6. The T-E-ordering L is called a *direct fuzzification* of a crisp ordering $\preceq$ if and only if it admits the following resolution:

$$L(x,y) = \begin{cases} 1 & \text{if } x \preceq y \\ E(x,y) & \text{otherwise} \end{cases}$$

It is worth to mention that strongly linear fuzzy orderings are uniquely characterized as direct fuzzifications of linear orderings.

Theorem 7. *Let L be a direct fuzzification of a crisp ordering $\preceq$. Then the following holds:*

$$\mathrm{ATL}(A) = \mathrm{EXT}(\mathrm{LTR}(A)) = \mathrm{LTR}(\mathrm{EXT}(A)) = \mathrm{EXT}(A) \cup \mathrm{LTR}(A) \quad (4)$$

$$\mathrm{ATM}(A) = \mathrm{EXT}(\mathrm{RTL}(A)) = \mathrm{RTL}(\mathrm{EXT}(A)) = \mathrm{EXT}(A) \cup \mathrm{RTL}(A) \quad (5)$$

The operator LTR *denotes the hull with respect to $\preceq$ while* RTL *stands for the hull operator with respect to the inverse relation of $\preceq$:*

$$\mu_{\mathrm{LTR}(A)}(x) = \sup\{\mu_A(y) \mid y \preceq x\}$$
$$\mu_{\mathrm{RTL}(A)}(x) = \sup\{\mu_A(y) \mid x \preceq y\}$$

Moreover, $\mathrm{ATL}(A)$ *is the smallest superset of A which is extensional and has a non-decreasing membership function. Analogously,* $\mathrm{ATM}(A)$ *is the smallest superset of A which is extensional and has a non-increasing membership function.*

$$\begin{array}{ccc} A & \xrightarrow{\text{EXT}} & \text{EXT}(A) \\ {\scriptstyle \text{LTR}}\big\downarrow & & {\scriptstyle \text{LTR}}\big\downarrow \\ \text{LTR}(A) & \xrightarrow{\text{EXT}} & \text{ATL}(A) \end{array}$$

Figure 1: A commutative diagram depicting the relationship (4) for a given fuzzy set A.

The correspondences (4) and (5) can be interpreted as a commutative diagram which is visualized in Figure 1.

In Proposition 5, we have already clarified the convexity of $\text{ATL}(A)$ and $\text{ATM}(A)$. Now let us turn to a deeper investigation of convexity in this context.

Lemma 8. *Assume that $\preceq$ is an arbitrary, not necessarily linear ordering. Then the fuzzy set*

$$\text{CVX}(A) = \text{LTR}(A) \cap \text{RTL}(A)$$

is the smallest convex superset of A.

Theorem 9. *With the assumptions of Theorem 7 and the definition*

$$\text{ECX}(A) = \text{ATL}(A) \cap \text{ATM}(A),$$

the following representation holds:

$$\text{ECX}(A) = \text{EXT}(\text{CVX}(A)) = \text{CVX}(\text{EXT}(A)) = \text{EXT}(A) \cup \text{CVX}(A)$$

Furthermore, $\text{ECX}(A)$ *is the smallest superset of A which is extensional and convex.*

Figure 2 shows an simple example demonstrating the actual meaning of the operators ATL and ATM as well as the correspondences of Theorem 7:

$$E(x,y) = \max(1 - |x-y|, 0)$$

$$L(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ \max(1-x+y, 0) & \text{otherwise} \end{cases}$$

One easily verifies that E is, indeed, a $T_\mathbf{L}$-equivalence on the real numbers and that L is a $T_\mathbf{L}$-E-ordering, which directly fuzzifies the natural ordering of real numbers $\leq$, where $T_\mathbf{L}$ stands for the Łukasiewicz t-norm

$$\max(x+y-1, 0).$$

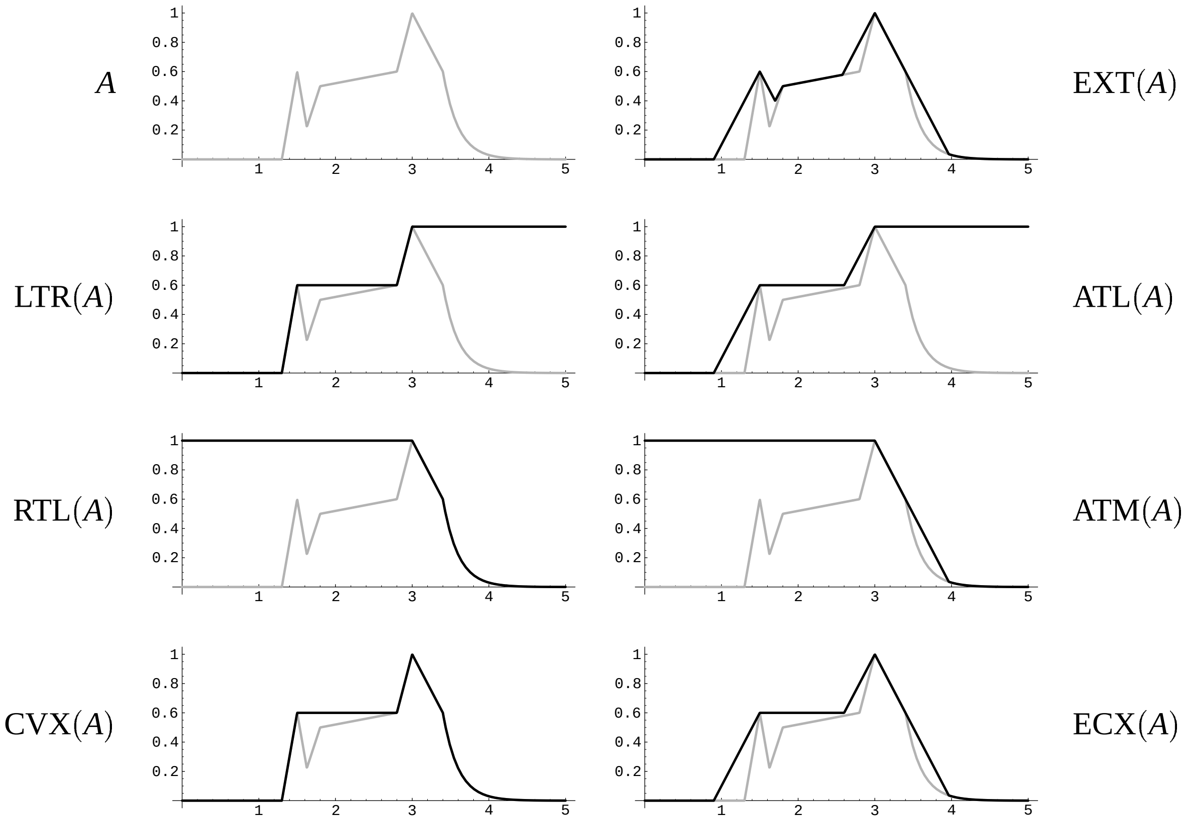

Figure 2: A fuzzy set $A \in \mathcal{F}(\mathbb{R})$ and the results which are obtained when applying various ordering-based hull operators.

4 More Ordering-Based Operators

The previous discussions enable us to define some other ordering-based operators which could be useful in applications. In the following, assume that L is a T-E-ordering on the domain X.

Definition 10. Let A be an arbitrary fuzzy subset of X. Then we can define the following unary modifiers:

1. \`Strictly greater than A' (SGT(A)):

$$\mathrm{SGT}(A) = \mathrm{ATL}(A) \cap \complement\mathrm{ECX}(A)$$

2. \`Strictly less than A' (SLS(A)):

$$\mathrm{SLS}(A) = \mathrm{ATM}(A) \cap \complement\mathrm{ECX}(A)$$

3. \`Within A' (WIT(A)):

$$\mathrm{WIT}(A) = \mathrm{ECX}(A) \cap \complement\mathrm{EXT}(A)$$

Note that $\mathrm{SGT}(A)$ does not necessarily coincide with the hull of A with respect to the dual relation of L. The same applies to $\mathrm{SLS}(A)$ and the complement relation of L. The operator WIT provides a method for extracting "holes" in non-convex fuzzy sets, where, obviously, $\mathrm{WIT}(A)$ is empty if A is convex.

Definition 11. For two fuzzy subsets $A,B \in \mathcal{F}(X)$, we can define the following two binary modifiers:

1. `Extensional convex closure of A and B' ($\mathrm{ECL}(A,B)$):

$$\mathrm{ECL}(A,B) = \mathrm{ECX}(A \cup B)$$

2. `Between A and B' ($\mathrm{BTW}(A,B)$):

$$\mathrm{BTW}(A,B) = \mathrm{WIT}(A \cup B)$$

5 Conclusion

This paper shows a constructive and intuitive way to define general concepts of fuzzy *`at least'* and *`at most'*. These two basic operations can be used easily to define other ordering-based modifiers and connectives, such as, *`between'*.

We have seen in which areas these modifiers and connectives can be useful. It remains to clarify the properties of these operators in connection with fuzzy inference. In particular, one may argue that a substitution of rules, as in the example in Section 1, does not necessarily result in the same input-output behavior. Hence, it is desirable, on the one hand, to study sufficient conditions under which the substitution of rules results in the same input-output behavior and, on the other hand, to study how drastic these changes really are in analytical terms.

References

[1] BODENHOFER, U. *A Similarity-Based Generalization of Fuzzy Orderings*. PhD thesis, Johannes Kepler Universität Linz, 1998.

[2] GOTTWALD, S. *Fuzzy Sets and Fuzzy Logic*. Vieweg, Braunschweig, 1993.

[3] HÁJEK, P. *Metamathematics of Fuzzy Logic*. Kluwer Academic Publishers, Dordrecht, 1998.

[4] KLAWONN, F., AND CASTRO, J. L. Similarity in fuzzy reasoning. *Mathware Soft Comput. 3*, 2 (1995), 197–228.

[5] KÓCZY, L. T., AND HIROTA, K. Ordering, distance and closeness of fuzzy sets. *Fuzzy Sets and Systems 59*, 3 (1993), 281–293.

[6] KÓCZY, L. T., AND HIROTA, K. Size reduction by interpolation in fuzzy rule bases. *IEEE Trans. Syst. Man Cybern. 27*, 1 (1997), 14–25.

[7] KRUSE, R., GEBHARDT, J., AND KLAWONN, F. *Foundations of Fuzzy Systems*. John Wiley & Sons, New York, 1994.

[8] LOWEN, R. Convex fuzzy sets. *Fuzzy Sets and Systems 3* (1980), 291–310.

[9] VALVERDE, L. On the structure of F-indistinguishability operators. *Fuzzy Sets and Systems 17*, 3 (1985), 313–328.

[10] ZADEH, L. A. Fuzzy sets. *Inf. Control 8* (1965), 338–353.

[11] ZADEH, L. A. Similarity relations and fuzzy orderings. *Inform. Sci. 3* (1971), 177–200.

[12] ZADEH, L. A. Outline of a new approach to the analysis of complex systems and decision processes. *IEEE Trans. Syst. Man Cybern. 3*, 1 (1973), 28–44.