Mathware & Soft Computing 15 (2008) 201-218

Orderings of Fuzzy Sets Based on Fuzzy Orderings Part I: The Basic Approach

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Abstract

The aim of this paper is to present a general framework for comparing fuzzy sets with respect to a general class of fuzzy orderings. This approach includes known techniques based on generalizing the crisp linear ordering of real numbers by means of the extension principle, however, in its general form, it is applicable to any fuzzy subsets of any kind of universe for which a fuzzy ordering is known—no matter whether linear or partial.

I.1 Introduction

There is no doubt that orderings and rankings are essential in every field related to decision making. Admitting vagueness or impreciseness naturally results in the need for specifying vague preferences in crisp domains, but also in the demand for a framework in which it is even possible to decide between fuzzy alternatives. It is, therefore, not surprising that orderings and rankings of fuzzy sets have become main objects of study in fuzzy decision analysis and related disciplines.

Even though only seldom recognized, orderings of fuzzy sets are also important in areas related to fuzzy systems and fuzzy control, where the ordering of a numerical domain is most often used when defining fuzzy sets—there might only be a minority of fuzzy systems or controllers in which ordinal expressions like "small", "medium", or "large" do not occur. The inherent ordering of such expressions is particularly crucial if automatic tuning procedures are concerned which are supposed to give *interpretable*, i.e. understandable, results [7, 8, 14]. Similar questions arise in *linguistic approximation* [26, 42] which may be considered as a kind of inverse procedure—finding a linguistic label for a given fuzzy set. A third application scenario is *rule interpolation* [37, 38], which is concerned with obtaining conclusions for observations that are not covered by any antecedent in a fuzzy rule base. Then orderings of fuzzy sets are able to provide criteria for determining between which rules the interpolation should take place [37].

There is one more emerging domain in which orderings of fuzzy sets are supposed to play a central role. Type-2 fuzzy sets were introduced more than thirty years ago [51], but only in the last ten years they have attracted the interest of a larger research community. Most applications of type-2 fuzzy sets are rather pragmatically oriented and simply use the extension principle [51] to extend well-known fuzzy logical operations like t-norms to type-2 fuzzy sets. However, similar to research efforts on interval-valued/Atanassov's intuitionistic fuzzy sets [21], axiomatic approaches to operations on type-2 fuzzy sets are slowly emerging [46]. Orderings of fuzzy sets are indispensable for defining monotonicity of operations on type-2 fuzzy sets. The compatibility with the extension principle is then imperative to maintain a meaningful connection with existing approaches to define operations on type-2 fuzzy sets.

Since the 1970s, a host of different methods for ordering or ranking fuzzy sets has been published [1, 2, 12, 15, 16, 17, 18, 19, 20, 23, 25, 27, 28, 33, 39, 40, 43, 44, 45, 49] (see [11, 47, 48] for detailed reviews). In order to provide profound motivation for adding yet another approach, let us review some common characteristics of these methods:

1. All methods are defined for so-called *fuzzy quantities*—fuzzy subsets of the real numbers. Not only from the theoretical, but also from the practical point of view, it could be interesting to consider arbitrary ordered domains, without any restriction in terms of the underlying domain or linearity of the ordering.

2. The applicability of many ordering methods is restricted to fuzzy quantities having special properties, such as convexity, normality or continuity [22], where *fuzzy numbers* are the most important sub-class which is considered in a large part of literature. Some authors restrict to such sub-classes, because their only motivation is to rank/order imprecise measurements—without the need of more general structures. Other authors, however, make such restrictions to guarantee desirable properties, for example, antisymmetry.

3. As long as linguistic expressions are represented by fuzzy subsets of numerical domains, there is a certain context-dependent notion of indistinguishability. It is worthwhile to take this indistinguishability into account, since not only the ranking of alternatives itself, but also the information that the difference between two alternatives is more or less negligible could be of interest. Almost all existing methods, however, do not offer the opportunity to integrate indistinguishability—often leading to "artificial preciseness"; and those approaches that incorporate indistinguishability in some way (e.g. [13, 28, 29]) are often limited to fuzzy quantities/numbers and employ rather restrictive distance-based notions of indistinguishability.

The purpose of this paper is to introduce and investigate an ordering method for arbitrary fuzzy subsets of an arbitrary (fuzzy) ordered domain that also takes a predefined general notion of indistinguishability into account. Unlike an earlier publication on this approach [4], this paper gives an up-to-date self-contained overview with full proofs and extensive examples.

After a review of preliminaries of fuzzy orderings and corresponding ordering-based modifiers in Section I.2, we construct the generalized approach starting from the usual ordering of real intervals in Section I.3, also establishing a link to the extension principle, and providing an investigation of its properties. Section I.4 highlights the limitations of the approach presented in this paper. Approaches to overcoming these limitations will be presented in Part II of this paper [6].

I.2 Preliminaries

In this paper, the letter X stands for an arbitrary but fixed non-empty set. Uppercase letters will be used synonymously for denoting fuzzy sets and their corresponding membership functions. The symbol $\mathcal{F}(X)$ denotes the *fuzzy powerset* of X. The *height* of a fuzzy set A is defined as $\text{height}(A) = \sup\{A(x) \mid x \in X\}$. We call a fuzzy set A *normalized* if $\text{height}(A) = 1$ and *normal* if there exists an $x \in X$ such that $A(x) = 1$. Furthermore, for $\alpha \in [0, 1[$, let $[A]_{\underline{\alpha}} = \{x \in X \mid A(x) > \alpha\}$ denote the *strict α-cut* of A. The relation $A \subseteq B$, as usual, denotes the crisp inclusion of fuzzy sets (i.e. $A \subseteq B$ if and only if $A(x) \leq B(x)$ for all $x \in X$ [50]). $A \cap B$ and $A \cup B$ stand for the intersection and union of two fuzzy sets A and B with respect to minimum and maximum, respectively. Moreover, the reader is assumed to be familiar with the basics of triangular norms [36]. Throughout the whole text, the symbol T is supposed to denote an arbitrary but fixed left-continuous t-norm.

Since, according to the above discussions, the ordering method should be able to cope with vagueness and indistinguishability, all studies in this paper will be based on the similarity-based definition of fuzzy orderings (for extensive studies, see [3, 5, 31]; for comprehensive reference to fuzzy equivalence relations, see e.g. [10]). We only recall the very basic definitions.

Definition I.1. A fuzzy relation $E : X^2 \to [0, 1]$ is called *fuzzy equivalence relation* with respect to T, for brevity *T-equivalence*, if the following three axioms are fulfilled for all $x, y, z \in X$:

(i)	Reflexivity:	$E(x, x) = 1$
(ii)	Symmetry:	$E(x, y) = E(y, x)$
(iii)	T-transitivity:	$T(E(x, y), E(y, z)) \leq E(x, z)$

Definition I.2. Consider a T-equivalence $E : X^2 \to [0, 1]$. A fuzzy relation $L : X^2 \to [0, 1]$ is called *fuzzy ordering* with respect to T and E, for brevity *T-E-ordering*, if it fulfills the following three axioms for all $x, y, z \in X$:

(i)	E-reflexivity:	$E(x, y) \leq L(x, y)$
(ii)	T-E-antisymmetry:	$T(L(x, y), L(y, x)) \leq E(x, y)$
(iii)	T-transitivity:	$T(L(x, y), L(y, z)) \leq L(x, z)$

A subclass that will be of special importance in the following are so-called direct fuzzifications.

Definition I.3. Consider a T-equivalence $E : X^2 \to [0, 1]$ and a crisp ordering $\preceq$ on X. Then a T-E-ordering $L : X^2 \to [0, 1]$ is called a *direct fuzzification* of $\preceq$ if the following representation holds for all $x, y \in X$:

$$L(x, y) = \begin{cases} 1 & \text{if } x \preceq y \\ E(x, y) & \text{otherwise} \end{cases}$$

It is worth to mention that there is a one-to-one correspondence between direct fuzzifications of crisp linear orderings and fuzzy orderings L that additionally fulfill *strong completeness* [3], i.e. $\max(L(x,y), L(y,x)) = 1$ for all $x, y \in X$. Such fuzzy orderings can be interpreted as linear orderings with imprecision [5].

The modifiers 'at least' and 'at most' with respect to a fuzzy ordering will be essential for all further investigations. They can be defined using the direct image of the fuzzy ordering [30] and can be understood as hulls/closures [9].

Definition I.4. Consider a T-equivalence $E : X^2 \to [0,1]$ and a T-E-ordering $L : X^2 \to [0,1]$. Then, for a given fuzzy set $A \in \mathcal{F}(X)$, the fuzzy sets 'at least A' and 'at most A' (with respect to L), abbreviated $\mathrm{ATL}(A)$ and $\mathrm{ATM}(A)$, respectively, are defined as follows (for all $x \in X$):

$$\mathrm{ATL}(A)(x) = \sup\{T(A(y), L(y,x)) \mid y \in X\}$$
$$\mathrm{ATM}(A)(x) = \sup\{T(A(y), L(x,y)) \mid y \in X\}$$

The well-known *extensional hull* operator [34, 35] will be denoted EXT in the following, i.e., for all $x \in X$,

$$\mathrm{EXT}(A)(x) = \sup\{T(A(y), E(y,x)) \mid y \in X\} \;.$$

If L is a crisp ordering, i.e. $L(x,y) \in \{0,1\}$ for all $x, y \in X$, we will often work with infix symbols, e.g. $\preceq$. In case that we want to make explicit that the underlying ordering is crisp, we will use the notations $\mathrm{LTR}(A)$ and $\mathrm{RTL}(A)$ (standing for left-to-right and right-to-left closure) instead of $\mathrm{ATL}(A)$ and $\mathrm{ATM}(A)$, respectively:

$$\mathrm{LTR}(A)(x) = \sup\{A(y) \mid y \in X \,\&\, y \preceq x\}$$
$$\mathrm{RTL}(A)(x) = \sup\{A(y) \mid y \in X \,\&\, x \preceq y\}$$

It is easy to see that $\mathrm{LTR}(A)$ is the smallest fuzzy superset of A that has a non-decreasing membership function (with respect to $\preceq$) and that $\mathrm{RTL}(A)$ is the smallest fuzzy superset of A that has a non-increasing membership function.

In direct analogy, $\mathrm{ATL}(A)$ can be regarded as the smallest fuzzy superset of A the membership function of which is non-decreasing with respect to L—even if L is not crisp; analogously for $\mathrm{ATM}(A)$. The following fundamental theorem supports this viewpoint (for a more detailed argumentation, see [9]). It shows that, in case that L is a direct fuzzification of a crisp ordering $\preceq$, the operators LTR and RTL commute with the extensional hull operator EXT and the resulting compound operators coincide with ATL and ATM, respectively. This fact will have central importance in the following.

Theorem I.5. [9] *Consider a T-equivalence $E : X^2 \to [0,1]$, a crisp ordering $\preceq$, and a T-E-ordering $L : X^2 \to [0,1]$ which is a direct fuzzification of $\preceq$. Then the following holds for all fuzzy sets $A \in \mathcal{F}(X)$:*

$$\mathrm{ATL}(A) = \mathrm{LTR}(\mathrm{EXT}(A)) = \mathrm{EXT}(\mathrm{LTR}(A)) = \mathrm{EXT}(A) \cup \mathrm{LTR}(A)$$
$$\mathrm{ATM}(A) = \mathrm{RTL}(\mathrm{EXT}(A)) = \mathrm{EXT}(\mathrm{RTL}(A)) = \mathrm{EXT}(A) \cup \mathrm{RTL}(A)$$

Convexity of fuzzy sets and convex hulls will also be relevant in the remaining paper. In contrast to vector space-based definitions of convexity [41, 50], we will use a general ordering-based definition of convexity. It is easy to check that, for the real numbers and their natural linear ordering, the two definitions are equivalent.

Definition I.6. Consider a crisp ordering $\preceq$ on X. Then a fuzzy set $A \in \mathcal{F}(X)$ is called *convex* (with respect to $\preceq$) if the following holds for all $x, y, z \in X$:

$$x \preceq y \preceq z \;\Rightarrow\; A(y) \geq \min(A(x), A(z))$$

Proposition I.7. [9] *Consider a crisp ordering $\preceq$ on X. Then the smallest convex fuzzy superset of a given fuzzy set $A \in \mathcal{F}(X)$, the so-called* convex hull *of A, is uniquely given as*

$$\mathrm{CVX}(A) = \mathrm{LTR}(A) \cap \mathrm{RTL}(A).$$

The convex hull defined above can be generalized to arbitrary fuzzy orderings in a straightforward way.

Definition I.8. Consider a T-equivalence $E : X^2 \to [0,1]$ and a T-E-ordering $L : X^2 \to [0,1]$. Then the operator ECX is defined as follows (for all $A \in \mathcal{F}(X)$):

$$\mathrm{ECX}(A) = \mathrm{ATL}(A) \cap \mathrm{ATM}(A)$$

The question is whether $\mathrm{ECX}(A)$ can also be considered as some kind of convex hull (with respect to L). If a fuzzy ordering L is a direct fuzzification of a crisp ordering $\preceq$, we obtain that the convex hull CVX also commutes with the extensional hull EXT and that the compound operator is nothing else but ECX (similar to Theorem I.5).

Theorem I.9. [9] *Consider a T-equivalence $E : X^2 \to [0,1]$, a crisp ordering $\preceq$, and a T-E-ordering $L : X^2 \to [0,1]$ which is a direct fuzzification of $\preceq$. Then the following equality holds for all $A \in \mathcal{F}(X)$:*

$$\mathrm{ECX}(A) = \mathrm{CVX}(\mathrm{EXT}(A)) = \mathrm{EXT}(\mathrm{CVX}(A)) = \mathrm{EXT}(A) \cup \mathrm{CVX}(A)$$

Almost needless to mention, the two operators CVX and ECX coincide if L is a crisp ordering. Theorem I.9, therefore, justifies the viewpoint to consider $\mathrm{ECX}(A)$ a kind of generalized convex hull with respect to a fuzzy ordering L. In the following, we will call it *extensional convex hull* of A.

Example I.10. Figure 1 shows a simple example demonstrating the actual meanings of the operators ATL, ATM, and $\mathrm{ECX}(A)$ as well as the correspondences of Theorems I.5 and I.9. We consider the following two fuzzy relations on the real numbers:

$$E(x, y) = \max(1 - |x - y|, 0) \qquad L(x, y) = \max(\min(1 - x + y, 1), 0)$$

One easily verifies that E is a $T_{\mathbf{L}}$-equivalence and that L is a $T_{\mathbf{L}}$-E-ordering that directly fuzzifies the natural linear ordering of real numbers, where $T_{\mathbf{L}}$ stands for the Łukasiewicz t-norm $T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$.

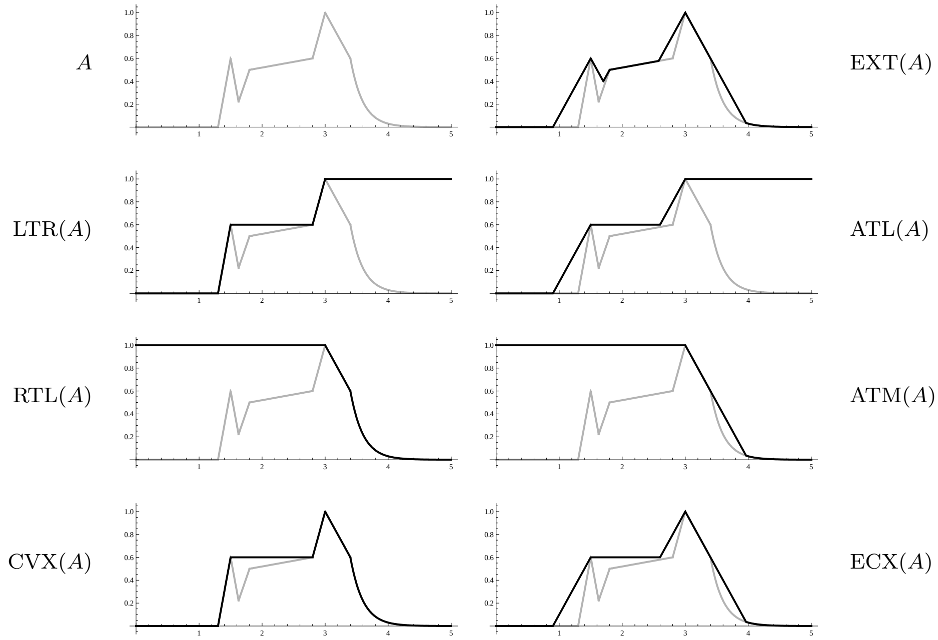

Figure 1: A fuzzy quantity A and the results that are obtained when applying various ordering-based operators.

I.3 The Basic Approach and Its Properties

In order to have a clear motivation, let us start from a well-known ordering procedure for real intervals (with respect to the usual ordering of real numbers):

$$[a,b] \leq_I [c,d] \;\Leftrightarrow\; a \leq c \,\&\, b \leq d \tag{1}$$

It is easy to check that $\leq_I$ is a partial ordering. The inequality $a \leq c$ means that there are no elements of the set $[c,d]$ that are below the entire interval $[a,b]$. The inequality $b \leq d$, analogously, means that there are no elements of $[a,b]$ that lie completely above $[c,d]$. This criterion can be generalized to arbitrary crisp subsets of an ordered set $(X, \preceq)$ as follows:

$$M \preceq_I N \;\Leftrightarrow\; \big((\forall x \in N)(\exists y \in M)\ y \preceq x\big) \,\&\, \big((\forall x \in M)(\exists y \in N)\ x \preceq y\big)$$

The following lemma provides an equivalent formulation by means of the operators LTR and RTL which will be the basis of all our further generalizations.

Lemma I.11. *Consider a crisp ordering $\preceq$ on X. Then the following equivalences hold for all $M, N \subseteq X$:*

$$\begin{aligned} \mathrm{LTR}(M) \supseteq \mathrm{LTR}(N) \;&\Leftrightarrow\; (\forall x \in N)(\exists y \in M)\ y \preceq x \\ \mathrm{RTL}(M) \subseteq \mathrm{RTL}(N) \;&\Leftrightarrow\; (\forall x \in M)(\exists y \in N)\ x \preceq y \end{aligned}$$

Furthermore, $M \preceq_I N$ *holds if and only if the following assertion is fulfilled:*

$$\mathrm{LTR}(M) \supseteq \mathrm{LTR}(N) \,\&\, \mathrm{RTL}(M) \subseteq \mathrm{RTL}(N) \tag{2}$$

Proof. It is easy to see that, for a crisp set M and a crisp ordering $\preceq$, the definition of $\mathrm{LTR}(M)$ simplifies to

$$\mathrm{LTR}(M) = \{x \in X \mid (\exists y \in M)\ y \preceq x\}.$$

Then $N \subseteq \mathrm{LTR}(M)$ is equivalent to

$$(\forall x \in N)(\exists y \in M)\ y \preceq x.$$

As LTR is a non-decreasing operator with respect to inclusion (i.e. $M \subseteq N$ implies $\mathrm{LTR}(M) \subseteq \mathrm{LTR}(N)$) and idempotent (i.e. $\mathrm{LTR}(\mathrm{LTR}(M)) = \mathrm{LTR}(M)$) [9], the following holds:

$$N \subseteq \mathrm{LTR}(M) \;\Leftrightarrow\; \mathrm{LTR}(N) \subseteq \mathrm{LTR}(M)$$

So, we have proven the following equivalence:

$$\mathrm{LTR}(M) \supseteq \mathrm{LTR}(N) \;\Leftrightarrow\; (\forall x \in N)(\exists y \in M)\ y \preceq x$$

Applying analogous arguments, we can prove

$$\mathrm{RTL}(M) \subseteq \mathrm{RTL}(N) \;\Leftrightarrow\; \big(\forall x \in M)(\exists y \in N)\ x \preceq y\big).$$

These two equivalences, of course, imply that $M \preceq_I N$ holds if and only if (2) holds. □

Since the operators LTR and RTL are not restricted to crisp sets, we can write down a generalization of $\preceq_I$ to fuzzy sets immediately.

Definition I.12. Consider a crisp ordering $\preceq$ on X. We define the following binary relation for $A, B \in \mathcal{F}(X)$:

$$A \preceq_I B \;\Leftrightarrow\; \big(\mathrm{LTR}(A) \supseteq \mathrm{LTR}(B) \;\&\; \mathrm{RTL}(A) \subseteq \mathrm{RTL}(B)\big)$$

This means that we are able to order fuzzy sets with respect to a crisp ordering $\preceq$. Recalling the interval ordering $\leq_I$, the inclusion $\mathrm{LTR}(A) \supseteq \mathrm{LTR}(B)$ corresponds to the fact that the left flank of A is to the left of the left flank of B (analogous to the inequality $a \leq b$), while the inclusion $\mathrm{RTL}(A) \subseteq \mathrm{RTL}(B)$ corresponds to the fact that the right flank of A is to the left of the right flank of B (analogous to the inequality $c \leq d$).

Alternatively, we could have extended $\preceq_I$ to fuzzy sets using the *extension principle* [51], i.e. by applying the ordering $\preceq_I$ to each strict α-cut. The following proposition demonstrates that the two ways are equivalent, i.e. Definition I.12 is fully consistent with the extension principle.

Proposition I.13. *Consider a crisp ordering* $\preceq$ *on* X*. Then, for all* $A, B \in \mathcal{F}(X)$*,* $A \preceq_I B$ *holds if and only if the following assertion is fulfilled:*

$$(\forall \alpha \in [0,1[)\ [A]_{\underline{\alpha}} \preceq_I [B]_{\underline{\alpha}}$$

Proof. It is trivial to see and well-known that the following holds for all $A, B \in \mathcal{F}(X)$:

$$A \subseteq B \;\Leftrightarrow\; (\forall \alpha \in [0,1[)\; [A]_{\underline{\alpha}} \subseteq [B]_{\underline{\alpha}}$$

Hence, we obtain that $\mathrm{LTR}(A) \supseteq \mathrm{LTR}(B) \,\&\, \mathrm{RTL}(A) \subseteq \mathrm{RTL}(B)$ holds if and only if, for all $\alpha \in [0,1[$,

$$[\mathrm{LTR}(A)]_{\underline{\alpha}} \supseteq [\mathrm{LTR}(B)]_{\underline{\alpha}} \,\&\, [\mathrm{RTL}(A)]_{\underline{\alpha}} \subseteq [\mathrm{RTL}(B)]_{\underline{\alpha}}. \tag{3}$$

Now we prove that LTR commutes with strict α-cuts:

$$\begin{aligned}
[\mathrm{LTR}(A)]_{\underline{\alpha}} &= \{x \in X \mid \mathrm{LTR}(A)(x) > \alpha\} \\
&= \{x \in X \mid \sup\{A(y) \mid y \preceq x\} > \alpha\} \\
&= \{x \in X \mid (\exists y \in X)\; (A(y) > \alpha \,\&\, y \preceq x)\} \\
&= \{x \in X \mid (\exists y \in [A]_{\underline{\alpha}})\; y \preceq x\} \\
&= \mathrm{LTR}([A]_{\underline{\alpha}})
\end{aligned}$$

The same procedure can be carried out to prove

$$[\mathrm{RTL}(A)]_{\underline{\alpha}} = \mathrm{RTL}([A]_{\underline{\alpha}}),$$

and we finally obtain that (3) is equivalent to

$$\mathrm{LTR}\left([A]_{\underline{\alpha}}\right) \supseteq \mathrm{LTR}\left([B]_{\underline{\alpha}}\right) \,\&\, \mathrm{RTL}\left([A]_{\underline{\alpha}}\right) \subseteq \mathrm{RTL}\left([B]_{\underline{\alpha}}\right)$$

for all $\alpha \in [0,1[$. This is nothing else but $[A]_{\underline{\alpha}} \preceq_I [B]_{\underline{\alpha}}$ for all $\alpha \in [0,1[$, and the proof is completed. □

It is a well-known approach to define an ordering procedure for *fuzzy numbers* (i.e. convex normal fuzzy quantities with bounded support) by generalizing the interval order $\leq_I$ to fuzzy numbers by means of the extension principle [22, 24, 32, 38]. Proposition I.13 shows that, in case $X = \mathbb{R}$ equipped with the natural linear ordering $\leq$, Definition I.12 coincides exactly with this extension principle-based ordering procedure. However, our approach is not limited to fuzzy sets having special properties and is not at all limited to linear orderings or real numbers. Moreover, the generalization to an arbitrary fuzzy ordering L—unlike the extension principle-based approach—is more than straightforward, as we see next.

Definition I.14. Consider a T-equivalence $E : X^2 \to [0,1]$ and a T-E-ordering $L : X^2 \to [0,1]$. Then the relation $\preceq_L$ is defined in the following way (for all $A, B \in \mathcal{F}(X)$):

$$A \preceq_L B \;\Leftrightarrow\; \left(\mathrm{ATL}(A) \supseteq \mathrm{ATL}(B) \;\&\; \mathrm{ATM}(A) \subseteq \mathrm{ATM}(B)\right)$$

Analogous to the remark stated above, the inclusion $\mathrm{ATL}(A) \supseteq \mathrm{ATL}(B)$ corresponds to the fact that the left flank of A is to the left of the left flank of B and the inclusion $\mathrm{ATM}(A) \subseteq \mathrm{ATM}(B)$ corresponds to the fact that the right flank of A is to the left of the right flank of B—however, now in a more general setting, where flanks are defined on the basis of an arbitrary fuzzy ordering L.

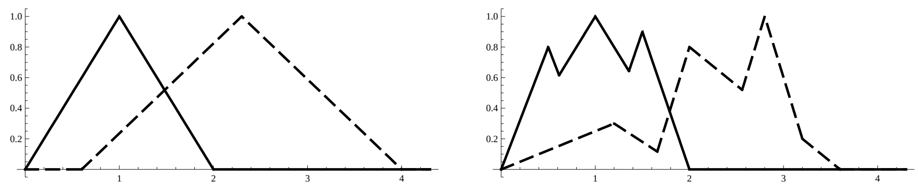

Figure 2: Left: two simple convex fuzzy quantities A_1 (solid line) and B_1 (dashed line) for which $A_1 \preceq_I B_1$ holds; right: two non-convex fuzzy quantities A_2 (solid line) and B_2 (dashed line) for which $A_2 \preceq_I B_2$ holds.

If L coincides with a crisp ordering $\preceq$, then $\preceq_L$ obviously coincides with $\preceq_I$. In case that L is a non-trivial fuzzy ordering that is a direct fuzzification, the two relations $\preceq_L$ and $\preceq_I$ do not coincide, but we can establish a clear link.

Proposition I.15. *Consider a T-equivalence $E : X^2 \to [0,1]$, a crisp ordering $\preceq$, and a T-E-ordering $L : X^2 \to [0,1]$ which is a direct fuzzification of $\preceq$. Then the following equivalence holds for all $A, B \in \mathcal{F}(X)$:*

$$A \preceq_L B \;\Leftrightarrow\; \mathrm{EXT}(A) \preceq_I \mathrm{EXT}(B)$$

Proof. Using the definition of $\preceq_L$ and Theorem I.5, we can infer

$$\begin{aligned} A \preceq_L B \;&\Leftrightarrow\; \big(\mathrm{ATL}(A) \supseteq \mathrm{ATL}(B) \;\&\; \mathrm{ATM}(A) \subseteq \mathrm{ATM}(B)\big) \\ &\Leftrightarrow\; \big(\mathrm{LTR}(\mathrm{EXT}(A)) \supseteq \mathrm{LTR}(\mathrm{EXT}(B)) \;\& \\ &\qquad \mathrm{RTL}(\mathrm{EXT}(A)) \subseteq \mathrm{RTL}(\mathrm{EXT}(B))\big), \end{aligned}$$

whereas the last assertion is obviously equivalent to $\mathrm{EXT}(A) \preceq_I \mathrm{EXT}(B)$. □

Example I.16. Consider the real numbers with the natural linear ordering. The left plot in Figure 2 shows two simple triangular fuzzy numbers A_1 and B_1 for which naturally $A_1 \preceq_I B_1$ holds (implying $A_1 \preceq_L B_1$ for any L directly fuzzifying the natural linear ordering of real numbers). The right plot in Figure 2 shows two non-convex fuzzy quantities A_2 and B_2. In this example, approaches assuming the convexity of the fuzzy sets are not applicable, although it is intuitively reasonable to assume that A_2 is, in some sense, smaller than B_2. In the proposed framework, $A_2 \preceq_I B_2$ actually holds (again implying $A_2 \preceq_L B_2$ for any L directly fuzzifying the natural linear ordering of real numbers).

Example I.17. The left plot in Figure 3 shows two triangular fuzzy numbers A_3 and B_3. Obviously, these two fuzzy quantities are incomparable with respect to $\preceq_I$ (if we use the linear ordering of real numbers again). Now let us consider the fuzzy ordering L (and its underlying $T_{\mathbf{L}}$-equivalence E) from Example I.10 instead of the crisp linear ordering and crisp equality of real numbers. The right plot in Figure 3 shows $\mathrm{EXT}(A_3)$ and $\mathrm{EXT}(B_3)$ (which coincides with B_3). Then it is obvious by Proposition I.15 that $A_3 \preceq_L B_3$. So we see that comparability is indeed a matter of the notion of similarity (i.e. the underlying fuzzy equivalence relation). The more tolerant the notion of similarity is, the easier two fuzzy sets are comparable.

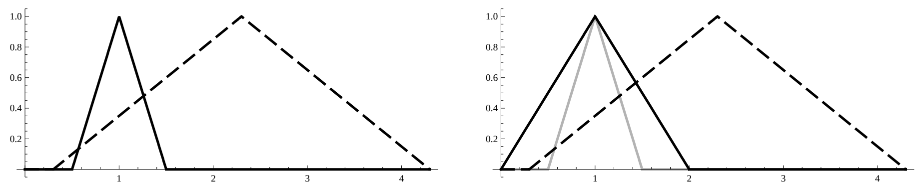

Figure 3: Left: two convex fuzzy quantities A_3 (solid line) and B_3 (dashed line) that are incomparable w.r.t. $\preceq_I$; right: the fuzzy quantities $EXT(A_3)$ (solid line; A_3 still plotted underneath in gray) and $\mathrm{EXT}(B_3) = B_3$ (dashed line).

Now we will provide a justification that the relation $\preceq_L$ (and $\preceq_I$ as well) may actually be considered an ordering of fuzzy sets.

Proposition I.18. *Consider a T-equivalence $E : X^2 \to [0,1]$ and a T-E-ordering $L : X^2 \to [0,1]$. Then the relation $\preceq_L$ is a preordering, i.e. it is reflexive and transitive.*

Proof. Follows directly from the fact that the inclusion of fuzzy sets $\subseteq$ is reflexive and transitive. □

The question remains whether the third fundamental axiom of orderings—antisymmetry—is fulfilled as well. The next result gives a unique characterization under which conditions antisymmetry is violated.

Theorem I.19. *Consider a T-equivalence $E : X^2 \to [0,1]$ and a T-E-ordering $L : X^2 \to [0,1]$. Then the following equivalence holds for all fuzzy sets $A, B \in \mathcal{F}(X)$:*

$$\big(A \preceq_L B \,\&\, A \succeq_L B\big) \;\Leftrightarrow\; \mathrm{ECX}(A) = \mathrm{ECX}(B)$$

Proof. The assertion $A \preceq_L B \,\&\, A \succeq_L B$ is trivially equivalent to $\mathrm{ATL}(A) = \mathrm{ATL}(B) \,\&\, \mathrm{ATM}(A) = \mathrm{ATM}(B)$, which obviously implies $\mathrm{ECX}(A) = \mathrm{ECX}(B)$. Therefore, it is sufficient to prove that $\mathrm{ECX}(A) = \mathrm{ECX}(B)$ implies $\mathrm{ATL}(A) = \mathrm{ATL}(B)$ and $\mathrm{ATM}(A) = \mathrm{ATM}(B)$. The operator ATL is non-decreasing with respect to inclusion [9]. Together with the trivial fact that $A \subseteq \mathrm{ECX}(A)$, we therefore obtain $\mathrm{ATL}(A) \subseteq \mathrm{ATL}(\mathrm{ECX}(A))$. On the other hand, $\mathrm{ECX}(A) \subseteq \mathrm{ATL}(A)$ holds trivially, which implies (using that fact that ATL is idempotent [9])

$$\mathrm{ATL}(\mathrm{ECX}(A)) \subseteq \mathrm{ATL}(\mathrm{ATL}(A)) = \mathrm{ATL}(A),$$

and we have proved that $\mathrm{ATL}(\mathrm{ECX}(A)) = \mathrm{ATL}(A)$. Hence, we obtain

$$\begin{aligned} \mathrm{ECX}(A) = \mathrm{ECX}(B) \;&\Rightarrow\; \mathrm{ATL}(\mathrm{ECX}(A)) = \mathrm{ATL}(\mathrm{ECX}(B)) \\ &\Rightarrow\; \mathrm{ATL}(A) = \mathrm{ATL}(B). \end{aligned}$$

Analogously, it is possible to show $\mathrm{ATM}(\mathrm{ECX}(A)) = \mathrm{ATM}(A)$, from which

$$\mathrm{ECX}(A) = \mathrm{ECX}(B) \;\Rightarrow\; \mathrm{ATM}(A) = \mathrm{ATM}(B)$$

follows, which completes the proof. □

Theorem I.19 states that the relation $\preceq_L$ cannot distinguish between fuzzy sets A, B if their extensional convex hulls $\mathrm{ECX}(A)$ and $\mathrm{ECX}(B)$ coincide. In case that L is a crisp ordering, the relation $\preceq_I$ is not able to distinguish between fuzzy sets A, B whose convex hulls $\mathrm{CVX}(A)$ and $\mathrm{CVX}(B)$ are the same. In case that L is a direct fuzzification, Theorem I.9 provides the link between these two cases.

The characterization of non-antisymmetry provided by Theorem I.19 gives us a hint what to do if we want to make $\preceq_L$ an ordering. The standard way to make a preordering an ordering is factorization with respect to the symmetric kernel of the preordering. In practice, however, it is often infeasible to work with factor sets. Theorem I.19 gives us a handy characterization of the symmetric kernel of $\preceq_L$. Therefore, we are not only able to do this factorization theoretically, but also to identify a correspondence between the factor set and a subclass of fuzzy sets having a special property. The following theorem provides the necessary details.

Theorem I.20. *Consider a T-equivalence $E : X^2 \to [0,1]$ and a T-E-ordering $L : X^2 \to [0,1]$. Then the binary relation*

$$A \cong_L B \ \Leftrightarrow\ \mathrm{ECX}(A) = \mathrm{ECX}(B)$$

is an equivalence relation on $\mathcal{F}(X)$. There is a one-to-one correspondence between the factor set $\mathcal{F}(X)_{/\cong_L}$ and the set of all extensional convex fuzzy subsets of X:

$$\mathcal{F}_L(X) = \{A \in \mathcal{F}(X) \mid A = \mathrm{ECX}(A)\}$$

The relation $\preceq_L$ is an ordering on $\mathcal{F}(X)_{/\cong_L}$ and $\mathcal{F}_L(X)$.

Proof. It is trivial that $\cong_L$ is an equivalence relation on $\mathcal{F}(X)$. Theorem I.19 implies that $\cong_L$ is nothing else but the symmetric kernel of $\preceq_L$. Now let us define the following mapping:

$$\begin{array}{rrcl} \mathbf{1}_L : & \mathcal{F}(X)_{/\cong_L} & \to & \mathcal{F}_L(X) \\ & \langle A \rangle & \mapsto & \mathrm{ECX}(A) \end{array}$$

$\mathbf{1}_L$ is well-defined, as an equivalence class $\langle A \rangle$ only contains fuzzy sets with the same extensional convex hulls. Clearly, $\mathbf{1}_L$ is injective—if we have fuzzy sets A, B for which $\langle A \rangle \neq \langle B \rangle$, it is immediate that $\mathrm{ECX}(A) \neq \mathrm{ECX}(B)$. $\mathbf{1}_L$ is surjective as well, as for every $B \in \mathcal{F}_L(X)$ (note that $\mathrm{ECX}(B) = B$ holds), there is an equivalence class $\langle A \rangle$ such that $\mathbf{1}_L(\langle A \rangle) = B$ (e.g. take $A = \mathrm{ECX}(B) = B$ itself). As $\cong_L$ is the symmetric kernel of $\preceq_L$, the projection of $\preceq_L$ to $\mathcal{F}(X)_{/\cong_L}$ is well-defined and antisymmetric, and so is the restriction of $\preceq_L$ to $\mathcal{F}_L(X)$. □

Remark I.21. If the underlying ordering relation L is crisp, Theorem I.20 can be rephrased as follows. The symmetric kernel of $\preceq_I$ is given as

$$A \cong_I B \ \Leftrightarrow\ \mathrm{CVX}(A) = \mathrm{CVX}(B).$$

There is a one-to-one correspondence between the factor set $\mathcal{F}(X)_{/\cong_I}$ and the set of all convex fuzzy subsets of X:

$$\mathcal{F}_I(X) = \{A \in \mathcal{F}(X) \mid A = \mathrm{CVX}(A)\}$$

The relation $\preceq_I$ is an ordering on $\mathcal{F}(X)_{/\cong_I}$ and $\mathcal{F}_I(X)$.

A consequence of Theorem I.20 is that $\preceq_L$ is an ordering if we restrict to $\mathcal{F}_L(X)$. This result still has a different quality if we compare our approach with the existing approaches that restricted to some special classes of fuzzy subsets (e.g. fuzzy numbers) in advance just to preserve properties like antisymmetry. The new framework presented in this paper is not restricted to a particular class of fuzzy sets. The preordering $\preceq_L$ can still distinguish between any two fuzzy sets as long as their extensional convex hulls do not coincide.

I.4 Limitations

Our initial objective was to define (pre)orderings of arbitrary fuzzy subsets of a domain for which a crisp or a fuzzy ordering is given. Now we should examine in detail whether this goal has really been achieved. We have found out that the relations considered so far are preorderings that violate antisymmetry only in the case that the (extensional) convex hulls of two fuzzy sets coincide. So, from a barely formal point of view, our initial requirements have been met. There are, however, still some weaknesses that are worth to be pointed out.

I.4.1 Crispness

The first remark concerns the way of comparing itself. Consider the two convex fuzzy quantities A_3 and B_3 from Example I.17 (see also the left plot in Figure 3). As pointed out in Example I.17, if we construct $\preceq_I$ by means of the natural linear ordering of real numbers, these two triangular fuzzy quantities are incomparable. No matter which fuzzy ordering L we consider, it is clear that A_3 is somehow “to the left” of B_3, but not fully. The question is whether it is really natural to compare vague phenomena crisply or if, as the example in Figure 3 suggests, this directly leads to what can be understood as “artificial preciseness”.

I.4.2 Different heights

One important feature of the ordering relations defined in this paper has not been mentioned yet: two fuzzy sets are in any case incomparable if they have different heights. The following lemma provides the basis for demonstrating this crucial fact.

Lemma I.22. *Consider a T-equivalence $E : X^2 \to [0,1]$ and a T-E-ordering $L : X^2 \to [0,1]$. Then the following equalities hold for all $A \in \mathcal{F}(X)$:*

$$\mathrm{height}(A) = \mathrm{height}(\mathrm{ATL}(A)) = \mathrm{height}(\mathrm{ATM}(A)) = \mathrm{height}(\mathrm{ECX}(A))$$

Proof. As $\mathrm{ATL}(A)$, $\mathrm{ATM}(A)$ and $\mathrm{ECX}(A)$ are all supersets of A, we trivially have

$$\mathrm{height}(A) \leq \min\big(\mathrm{height}(\mathrm{ATL}(A)), \mathrm{height}(\mathrm{ATM}(A)), \mathrm{height}(\mathrm{ECX}(A))\big).$$

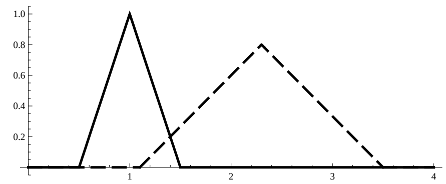

Figure 4: Two convex fuzzy quantities that seem to be in proper order with respect to the natural linear ordering of real numbers, but which are actually incomparable with respect to $\preceq_I$, since they have different heights.

From the trivial inequality $T(A(y), L(y,x)) \leq A(y)$, we can infer

$$\begin{aligned}\text{height}(\text{ATL}(A)) &= \sup_{x\in X} \text{ATL}(A)(x) = \sup_{x\in X} \sup_{y\in X} T(A(y), L(y,x)) \\ &\leq \sup_{y\in X} A(y) = \text{height}(A).\end{aligned}$$

Hence, we obtain $\text{height}(A) = \text{height}(\text{ATL}(A))$ (which already implies $\text{height}(A) = \text{height}(\text{ECX}(A))$, since $\text{ECX}(A)$ is a fuzzy subset of $\text{ATL}(A)$). Analogously, we can prove $\text{height}(A) = \text{height}(\text{ATM}(A))$. $\square$

As a special case, if we have a crisp ordering, we can infer the following from Lemma I.22:

$$\text{height}(A) = \text{height}(\text{LTR}(A)) = \text{height}(\text{RTL}(A)) = \text{height}(\text{CVX}(A))$$

Proposition I.23. *Consider a T-equivalence $E : X^2 \to [0,1]$ and a T-E-ordering $L : X^2 \to [0,1]$. Then the following implication holds for all $A, B \in \mathcal{F}(X)$:*

$$A \preceq_L B \;\Rightarrow\; \text{height}(A) = \text{height}(B)$$

Proof. Assume that $A \preceq_L B$ holds. From $\text{ATL}(A) \supseteq \text{ATL}(B)$, the inequality $\text{height}(\text{ATL}(A)) \geq \text{height}(\text{ATL}(B))$ follows trivially, which, by Lemma I.22, implies $\text{height}(A) \geq \text{height}(B)$. Analogously, we can infer $\text{height}(A) \leq \text{height}(B)$ from $\text{ATM}(A) \subseteq \text{ATM}(B)$, and $\text{height}(A) = \text{height}(B)$ follows. $\square$

As a consequence of Proposition I.23, we obtain that $\text{height}(A) \neq \text{height}(B)$ implies both $A \not\preceq_L B$ and $B \not\preceq_L A$. This means, as anticipated above, that two fuzzy sets with different heights are guaranteed to be incomparable. If the concrete setting/application does not justify to consider normal or normalized fuzzy sets only, we are left with the problem that two fuzzy sets may be incomparable although the order between them seems more than obvious (see Figure 4 for an example)—which is particularly counter-intuitive if the heights of the two fuzzy sets is almost, but not exactly, the same.

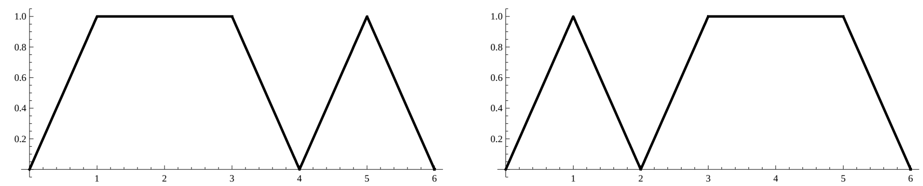

Figure 5: Two non-convex fuzzy quantities with equal convex hulls.

I.4.3 Non-antisymmetry

The third and last point of critique refers to (non-)antisymmetry. Theorem I.19 states that antisymmetry can be violated only if the (extensional) convex hulls of the two fuzzy sets coincide; Theorem I.20 shows that full antisymmetry can be maintained if we restrict ourselves to (extensional) convex fuzzy sets. One can observe, at least intuitively, that these results are not exhaustive. Consider, for instance, the two fuzzy quantities shown in Figure 5. Obviously, they have equal convex hulls, so the relation $\preceq_I$ (defined using the natural linear ordering of real numbers) cannot distinguish between them. The same happens with $\preceq_L$ if L is chosen as a direct fuzzification of the natural linear ordering of real numbers. Nevertheless, it makes sense to argue that the right fuzzy set should be ranked higher than the left one, simply looking at the positions of the two dents and the positions of the larger trapezoids.

I.5 Concluding Remarks

In this paper, a general method for ordering fuzzy sets with respect to fuzzy orderings has been introduced. If L is nothing else but the natural linear ordering of real numbers, the preordering $\preceq_I$ coincides with established extension principle-based approaches. The advantages of the framework presented here are (1) the possibility to deal with arbitrary (even partial) orderings, (2) the possibility to integrate fuzziness by considering fuzzy orderings, and (3) that the restriction to certain subclasses of fuzzy sets is not necessary in this approach. The arguments in Section I.4 underscore, however, that the most meaningful results are still obtained if fuzzy sets are considered that are normalized and convex. Generalizations that overcome these limitations will be presented in Part II of this paper [6].

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