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Ordinal Structures in Vague Environments

Habilitationsschrift

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To my happy little family,
my wife Verena Christina
and our baby son Lorenz Paul,
with love

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Introduction

It is an outstanding feature of human common sense thinking to inherently employ a gradual concept of equality with a tolerance for imprecision. Consider the notorious example how to define a set of tall people. Specifying a sharp limit, e.g. 180cm, leads to unnatural preciseness. While a person of 179.95cm would be classified as not tall, somebody of 180.05cm would be classified as tall, although it is virtually impossible to distinguish between the two, not even with measurements. Allowing a gradual transition of membership between the two classes “not tall” and “tall”—as fuzzy sets do—solves this problem in a simple and pragmatic, yet elegant and efficient, way. This viewpoint suggests that gradual similarity is, in some sense, an inherent component of fuzziness. The example also indicates that this kind of gradual similarity appears in ordinal contexts too, as the height of people and the concept “tall” have an ordinal structure.

Equivalence relations and orderings are Siamese twins in classical mathematics. Within the early gold rush of fuzzification of any classical mathematical concept, these two fundamental types of relations did not have to await the introduction of their fuzzy counterparts for a long time.

On the one hand, a well-established theory of *fuzzy equivalence relations* has been created within the last twenty years. It has turned out that fuzzy equivalence relations are most suitable concepts for modeling gradual similarity in the sense discussed above. The study of fuzzy equivalence relations has not only brought insight in a specific class of fuzzy relations, but also deepest insight in the inherent principles of fuzzy sets. Adopting the terminology of Klawonn et al., a domain equipped with some gradual, possibly implicit, concept of similarity modeled by a fuzzy equivalence relation is called a *vague environment*.

On the other hand, different concepts of *fuzzy orderings* have existed, but the theory remained underdeveloped. This is astonishing and seems almost paradoxical, as almost all fuzzy systems make implicit use of ordinal structures—there might be only a small minority of fuzzy systems in which expressions, such as “small”, “medium”, or “large’, do not occur. This was the starting point of a long-term research effort the outcome of which is subsumed in the present habilitation thesis. The ambitious long-term goal was not only to create a reasonable theory of fuzzy orderings, but also to achieve the integration of this concept into real-world applications. Consequently, this thesis is organized in three main parts that mark the three basic cornerstones on the way from theory to applications of ordinal structures in vague environments.

The problem of previous approaches to fuzzy orderings was the missing link to gradual similarity. Part I is devoted to the study of theoretical aspects of a new definition of fuzzy orderings that integrates gradual similarity into fuzzy orderings—thereby linking ordinal structures with vague environments. This proposal is accompanied by constructions, representations, and a thorough study of different concepts of linearity in the context of the generalized framework of fuzzy orderings, addressing most fundamental properties known from classical order theory.

Part II goes one step further in the direction of applications. Image operators of fuzzy orderings are not only of theoretical interest. They provide a means of defining ordering-based modifiers, that is, operators modeling the semantics of adverbs like "at least", "at most", and "between" with respect to a given fuzzy ordering—under the assumption that the semantics of the adjectives to which the adverb is applied to is modeled by a fuzzy set. As a byproduct, a general approach to orderings of fuzzy sets with respect to a fuzzy ordering was obtained which is most general in the sense that no specific prior assumption about the shape of the fuzzy sets or properties of the fuzzy ordering is necessary.

Part III is devoted to applications. The first paper addresses the applicability of fuzzy orderings in flexible query answering systems—an integration that seems natural and straightforward, but was missing so far because of the obvious lack of an appropriately expressive concept of fuzzy orderings. The four other papers are devoted to applications in fuzzy rule-based systems, ranging from a formal model of interpretability that makes use of orderings of fuzzy sets to fuzzy rule-based machine learning algorithms which, on the one hand, obey interpretability constraints and, on the other hand, employ ordering-based modifiers to improve compactness and expressiveness of the resulting rule systems.

Overview

Part I: Theory of Fuzzy Orderings

U. Bodenhofer. **A similarity-based generalization of fuzzy orderings preserving the classical axioms.** *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 8(5):593–610, 2000. World-Scientific, ISSN 0218-4885.

> This paper marks the first important milestone. It motivates the need for a new, more general approach to fuzzy orderings that takes a context of gradual similarity into account. This motivation is done by means of three case studies. Consequently, the generalized definition is presented along with basic properties and interpretations. It is elaborated in detail that the difficulties—according to the three case studies—are fully resolved in the new framework. As one of the main results, a fundamental representation theorem for strongly complete/linear fuzzy orderings is presented which had a strong influence on almost all other works that came after (Theorem 4.2).

U. Bodenhofer. **Representations and constructions of similarity-based fuzzy orderings.** *Fuzzy Sets and Systems*, 2003. (in press). Elsevier, ISSN 0165-0114.

> This paper continues the theoretical study of fuzzy orderings. The special emphasis of this paper is on construction and representation results. As the main result, the seamless integration of the new concept of fuzzy orderings into the existing theory of fuzzy relations is demonstrated. Firstly, the integration of fuzzy orderings into Valverde's fuzzy set-based representation is studied. Secondly, a general representation by means of generalized inclusion operators on the fuzzy power set is given.

U. Bodenhofer and F. Klawonn. **A formal study of linearity axioms for fuzzy orderings.** *Fuzzy Sets and Systems*. (accepted). Elsevier, ISSN 0165-0114.

In the theory of classical relations, linear orderings play an outstanding role. A fuzzy order theory would be meaningless without a proper concept of linearity. The definition of an axiom of linearity, however, is not so straightforward in the framework of fuzzy logics—definitions that are equivalent in Boolean logic have completely different meanings in fuzzy logics. This paper studies three established definitions with respect to most fundamental properties of linear orderings in the crisp case—the Szpilrajn Theorem, the intersection representation (i.e. that any partial ordering can be represented as an intersection of linear orderings), and the equivalence of linearity and maximality. The main result is that none of the three definitions preserves all three fundamental properties simultaneously if no special assumption about the underlying t-norm is made. It is, moreover, shown that no practically feasible axiom can be found such that the equivalence to maximality is maintained for any other t-norm than the minimum. The results, however, indicate that a residuum-based definition due to Höhle and Blanchard appears to be a reasonable compromise, at least for t-norms that induce a strong negation (including nilpotent t-norms).

Part II: Ordering-Based Modifiers and Orderings of Fuzzy Sets

U. Bodenhofer, M. De Cock, and E. E. Kerre. **Openings and closures of fuzzy preorderings: Theoretical basics and applications to fuzzy rule-based systems.** *Int. J. General Systems*. (accepted). Taylor & Francis, ISSN 0308-1079.

This paper has two aims. Firstly, the properties of image operators of fuzzy preorderings (i.e. reflexive and T-transitive fuzzy relations) is studied. This is done by generalizing existing results for fuzzy equivalence relations to the non-symmetric case. While this generalization is more or less a routine matter, a crucial new theorem about the commutativity of image operators is presented. In the second part, these results are applied to the specific case of fuzzy orderings, with the aim to define the two ordering-based modifiers “at least” and “at most” in a theoretically sound way. For the practically relevant case of direct fuzzifications (including strongly complete/linear fuzzy orderings), a representation theorem for the closure operator is given.

U. Bodenhofer. **A unified framework of opening and closure operators with respect to arbitrary fuzzy relations.** *Soft Computing*, 7:220–227, 2003. Springer, ISSN 1432-7643.

> This paper studies image operators of fuzzy relations in a more general setting without assuming T-transitivity. The result is a theoretically profound definition of opening and closure operators. It is shown that the existing opening and closure operators of fuzzy (pre)orderings smoothly fit into this more general framework. Moreover, it is demonstrated that even fuzzy morphological operators can be embedded.

U. Bodenhofer. **A general framework for ordering fuzzy sets.** In B. Bouchon-Meunier, J. Guitiérrez-Ríoz, L. Magdalena, and R. R. Yager, editors, *Technologies for Constructing Intelligent Systems 1: Tasks*, volume 89 of *Studies in Fuzziness and Soft Computing*, pages 213–224. Physica-Verlag, Heidelberg, 2002. ISBN 3-7908-1454-7.

> Orderings and rankings of fuzzy sets have been objects of study for a long time, resulting in a host of different approaches. This paper shows how a preordering of fuzzy sets can be defined by means of image operators of fuzzy orderings. The result is a general framework of orderings of fuzzy sets which, in contrast to earlier approaches, does not assume any specific class of fuzzy sets or any specific properties of the fuzzy ordering.

U. Bodenhofer. **Fuzzy “between” operators in the framework of fuzzy orderings.** In B. Bouchon-Meunier, L. Foulloy, and R. R. Yager, editors, *Intelligent Systems for Information Processing: From Representation to Applications*. Elsevier. (to appear).

> This paper continues the work on ordering-based modifiers by adding two fuzzy “between” operators, an inclusive and a non-inclusive one. These two operators are studied in detail. It is proved that the result of applying these operators to a pair of fuzzy sets (A, B) actually results in fuzzy sets that can be considered “between” the two arguments A and B. For this study, the general framework of orderings of fuzzy sets is employed.

Part III: Applications

U. Bodenhofer and J. Küng. **Fuzzy orderings in flexible query answering systems.** *Soft Computing*. (accepted). Springer, ISSN 1432-7643.

> The use of fuzzy equivalence relations has had a long tradition in flexible query answering systems (FQAS's), while fuzzy orderings have never been applied in this context. The reason was simply that previous approaches to fuzzy orderings were not rich enough to fulfill the needs of applications in FQAS's. The generalized definition of fuzzy orderings overcomes the shortcomings of previous approaches. Consequently, this paper describes the benefits of using the new approach of fuzzy orderings in flexible query answering systems in detail. As a case study, a simple and pragmatic variant of a flexible query answering system—the so-called Vague Query System (VQS)—is considered. The integration of fuzzy orderings into that system is provided in full detail along with the necessary theoretical background and examples.

U. Bodenhofer and P. Bauer. **A formal model of interpretability of linguistic variables.** In J. Casillas, O. Cordón, F. Herrera, and L. Magdalena, editors, *Interpretability Issues in Fuzzy Modeling*, Studies in Fuzziness and Soft Computing, pages 524–545. Springer, 2003. (in press).

> The commonly accepted key feature of fuzzy system is their interpretability, i.e. that a human expert can guess a fuzzy system's behavior, at least qualitatively. This property, however, is not satisfied by definition. In particular, if automatic tuning methods (e.g. fuzzy rule-based machine learning methods) are applied, non-interpretable black-box systems can be obtained that are fuzzy systems formally, but do not provide any insight for a human expert anymore. This paper studies which criteria are essential for interpretability. In these considerations, orderings and orderings of fuzzy sets play a crucial role. Finally, constraints for enforcing interpretability are presented that make use of ordinal information.

M. Drobics, U. Bodenhofer, and E. P. Klement. **FS-FOIL: An inductive learning method for extracting interpretable fuzzy descriptions.** *Internat. J. Approx. Reason.*, 32(2–3):131–152, 2003. Elsevier, ISSN 0888-613X.

> Quinlan's FOIL (First-Order Inductive Learner) method is a machine learning algorithm which tries to describe data samples that fulfill a given goal predicate by descriptions based on predicates defined for the input variables. In this way, a rule base of interpretable classification rules can be obtained. FS-FOIL is a highly efficient fuzzy variant of FOIL. Firstly, the use of fuzzy predicates bridges the gap between symbolic classification rules and numerical models with interpolative behavior. Secondly, the use of ordering-based modifiers and interpretability constraints guarantees that a compact set of highly interpretable rules is obtained.

M. Drobics, U. Bodenhofer, and W. Winiwarter. **Mining clusters and corresponding interpretable descriptions — a three-stage approach.** *Expert Systems*, 19(4):224–234, 2002. Blackwell Publishing, ISSN 0266-4720.

> This paper is devoted to descriptive cluster analysis from a new, holistic perspective. The proposed three-stage approach starts from the question how large, potentially incomplete and noisy, data sets can be clustered. Conventional crisp and fuzzy clustering algorithms are usually unable to handle very large data sets and usually require a considerable data quality to produce a meaningful and robust result. To compress and clean the data, the use of self-organizing maps is suggested as the first step. Then a fuzzy clustering algorithm can be applied. The third step is concerned with producing interpretable descriptions of the clusters. This is accomplished by regarding the cluster membership functions as goal predicates and employing FS-FOIL to generate the descriptions.

M. Drobics and U. Bodenhofer. **Fuzzy modeling with decision trees.** In *Proc. 2002 IEEE Int. Conf. on Systems, Man and Cybernetics*, pages 90–95, Hammamet, October 2002. ISBN 2-9512309-4-X.

> This paper generalizes another well-known machine learning algorithm—Quinlan's ID3 method for constructing decision trees. Analogously to FS-FOIL, the proposed FS-ID3 algorithm overcomes the discrepancy between symbolic classification rules and numerical models with interpolative behavior. FS-ID3 also integrates ordering-based modifiers and interpretability constraints.