Generalized Opening and Closure Operators of Fuzzy Relations

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Abstract

In this paper, we propose a generalized concept of openness and closedness with respect to arbitrary fuzzy relations, along with appropriate opening and closure operators. We will show that this framework includes the existing concept defined for fuzzy preorderings as well as the triangular norm-based approach to fuzzy mathematical morphology.

Keywords: closure, fuzzy mathematical morphology, fuzzy relation, opening.

1 Introduction

Opening and closure operators occur in at least two different contexts in fuzzy set theory. Firstly, it is possible to define meaningful concepts of opening and closure operators with respect to fuzzy preorderings. Secondly, such concepts are used in fuzzy mathematical morphology as well.

In analogy to topology, it seems reasonable to connect opening and closure operators to some concepts of openness and closedness. In a strict setting, the terms "opening" and "closure" only make sense if they give open and closed results, respectively, preferably fulfilling some extremal properties. Another basic requirement for the appropriateness of these terms is idempotency.

In the first case of fuzzy preorderings, all these basic properties can be satisfied [3, 4]. Unfortunately, all the properties collapse if either reflexivity or transitivity is not fulfilled, implying that these concepts are not applicable to general fuzzy relations.

Fuzzy mathematical morphology, on the other hand, uses slightly different concepts for defining opening and closure operators, for which some of the crucial properties we mentioned above can be guaranteed. A recent investigation [17] has shown that fuzzy mathematical morphology inherently uses concepts we know from the field of fuzzy relations, however, leaving the implications in terms of algebraic properties unclarified.

In this paper, we introduce general concepts of openness and closedness with respect to arbitrary fuzzy relations together with opening and closure operators which fulfill all necessary algebraic properties. It will turn out that this framework includes the fuzzy preordering-based concept as well as the t-norm-based approach to fuzzy mathematical morphology.

2 Preliminaries

Throughout the whole paper, we will not explicitly distinguish between fuzzy sets and their corresponding membership functions. Consequently, uppercase letters will be used for both synonymously.

We will restrict ourselves to common standard systems of fuzzy logics—logics on the unit interval equipped with a left-continuous t-norm and its corresponding residual implication [6, 7, 11, 14]. From the practical point of view, however, this is not a serious restriction.

Definition 1. A *triangular norm* (*t-norm* for short) is an associative, commutative, and non-decreasing binary operation on the unit interval (i.e. a $[0,1]^2 \to [0,1]$ mapping) which has 1 as neutral element. A t-norm is called *left-continuous* if and only if both partial mappings $T(x,.)$ and $T(.,x)$ are left-continuous.

In order to provide the reader with the basic properties of residual implications, let us briefly recall them. For proofs, the reader is referred to the literature (e.g. [6, 7]).

Definition 2. For a left-continuous t-norm T, the *residual implication (residuum)* $\vec{T}$ is defined as

$$\vec{T}(x,y) = \sup\{u \in [0,1] \mid T(u,x) \leq y\}.$$

Lemma 3. *Consider a left-continuous t-norm T. Then the following holds for all $x, y, z \in [0,1]$:*

1. $x \leq y \iff \vec{T}(x,y) = 1$
2. $T(x,y) \leq z \iff x \leq \vec{T}(y,z)$
3. $T\big(\vec{T}(x,y), \vec{T}(y,z)\big) \leq \vec{T}(x,z)$
4. $\vec{T}(1,y) = y$
5. $T\big(x, \vec{T}(x,y)\big) \leq y$
6. $y \leq \vec{T}\big(x, T(x,y)\big)$

Furthermore, $\vec{T}$ is non-increasing and left-continuous in the first argument and non-decreasing and right-continuous in the second argument.

In this paper, we will solely consider *binary fuzzy relations*, i.e. fuzzy subsets of a product space $X^2 = X \times X$, where X is an arbitrary crisp set, where images of fuzzy relations will be of particular importance.

Definition 4. Consider an arbitrary fuzzy subset $A \in \mathcal{F}(X)$. The *full image* of A under R, denoted $R{\uparrow}A$, and its dual $R{\downarrow}A$ are defined as

$$\begin{aligned} R{\downarrow}A(x) &= \inf_{y \in X} \vec{T}\big(R(x,y), A(y)\big), \\ R{\uparrow}A(x) &= \sup_{y \in X} T\big(A(y), R(y,x)\big). \end{aligned}$$

Lemma 5. *The following holds for all $A, B \in \mathcal{F}(X)$:*

1. $A \subseteq B \implies R{\downarrow}A \subseteq R{\downarrow}B$
2. $A \subseteq B \implies R{\uparrow}A \subseteq R{\uparrow}B$

Proof. These propositions follow directly from the monotonicity properties of triangular norms and their residual implications (see [6,9] for more detailed proofs). □

3 Opening and Closure Operators of Arbitrary Fuzzy Relations

We will now propose operators which enable us to define a concept of openness and closedness with respect to an arbitrary fuzzy relation. As it will turn out later, these operators directly correspond to the appropriate opening and closure operators. Throughout this section, assume that R is an arbitrary binary fuzzy relation on a domain X.

Definition 6. The operators R° and $R^\bullet$ are defined in the following way:

$$\begin{aligned} R^\circ A &= R{\uparrow}(R{\downarrow}A) \\ R^\bullet A &= R{\downarrow}(R{\uparrow}A) \end{aligned}$$

We will now investigate the properties of these two operators. Let us start with a following fundamental inclusion property.

Lemma 7. *The following chain of inclusions holds for any fuzzy set $A \in \mathcal{F}(X)$:*

$$R^\circ A \subseteq A \subseteq R^\bullet A$$

Proof. Consider an arbitrary $x \in X$:

$$\begin{aligned} R^\circ A(x) &= \sup_{y \in X} T\big(R{\downarrow}A(y), R(y,x)\big) \\ &= \sup_{y \in X} T\big(\inf_{z \in X} \vec{T}(R(y,z), A(z)), R(y,x)\big) \\ &= (*) \end{aligned}$$

Setting $z = x$, we obtain by Lemma 3, 5.,

$$\begin{aligned} (*) &\leq \sup_{y \in X} T\big(\vec{T}(R(y,x), A(x)), R(y,x)\big) \\ &= \sup_{y \in X} T\big(R(y,x), \vec{T}(R(y,x), A(x))\big) \\ &\leq A(x). \end{aligned}$$

For proving $A \subseteq R^\bullet A$, we apply an analogous technique (setting again $z = x$, but applying Prop. 6. of Lemma 3):

$$\begin{aligned} R^\bullet A(x) &= \inf_{y \in X} \vec{T}\big(R(x,y), R{\uparrow}A(y)\big) \\ &= \inf_{y \in X} \vec{T}\big(R(x,y), \sup_{z \in X} T(A(z), R(z,y))\big) \\ &\geq \inf_{y \in X} \vec{T}\big(R(x,y), T(R(x,y), A(x))\big) \\ &\geq A(x) \end{aligned}$$

□

We have already seen in Lemma 5 that the images $R\downarrow$ and $R\uparrow$ are monotonic. As we will see next, the same monotonicity trivially transfers to the two operators R° and $R^\bullet$.

Lemma 8. *The following holds for all $A, B \in \mathcal{F}(X)$:*

1. $A \subseteq B \implies R^\circ A \subseteq R^\circ B$
2. $A \subseteq B \implies R^\bullet A \subseteq R^\bullet B$

Proof. Immediate consequences of Lemma 5. $\square$

Now we are able to define the general concepts of openness and closedness.

Definition 9. A fuzzy set $A \in \mathcal{F}(X)$ is called R-*open* if and only if

$$R^\circ A = A.$$

Correspondingly, A is called R-*closed* if and only if

$$R^\bullet A = A.$$

The next theorem provides a unique characterization of R-openness and R-closedness by means of the two image operators $R\downarrow$ and $R\uparrow$, respectively.

Theorem 10. *The following equivalences hold for any fuzzy set $A \in \mathcal{F}(X)$:*

1. *A is R-open if and only if there exists a fuzzy set $B \in \mathcal{F}(X)$ such that $A = R\uparrow B$.*
2. *A is R-closed if and only if there exists a fuzzy set $C \in \mathcal{F}(X)$ such that $A = R\downarrow C$.*

Proof. 1. First of all, let us assume that A is R-open, i.e. $R\uparrow(R\downarrow A) = A$. Then choosing $B = R\downarrow A$ proves the first implication.

Now suppose that A can be represented as $R\uparrow B$ for some $B \in \mathcal{F}(X)$. We know from Lemma 7 that $A \supseteq R^\circ A$ in any case. In order to prove the reverse inclusion, let us consider the following (using Prop. 6. of Lemma 3):

$$\begin{aligned}
R\downarrow A(x) &= \inf_{y \in X} \vec{T}\big(R(x,y), A(y)\big) \\
&= \inf_{y \in X} \vec{T}\big(R(x,y), R\uparrow B(y)\big) \\
&= \inf_{y \in X} \vec{T}\big(R(x,y), \sup_{z \in X} T(B(z), R(z,y))\big) \\
&\geq \inf_{y \in X} \vec{T}\big(R(x,y), T(R(x,y), B(x))\big) \\
&\geq B(x)
\end{aligned}$$

We have shown that $R\downarrow A \supseteq B$; therefore, by Lemma 5,

$$R^\circ A = R\uparrow(R\downarrow A) \supseteq R\uparrow B = A,$$

which implies $R^\circ A = A$.

2. If we assume that A is R-closed, i.e. $R\downarrow(R\uparrow A) = A$, choosing $B = R\uparrow A$ proves the first implication.

Conversely, assume that A can be represented as $R\downarrow C$ for some $C \in \mathcal{F}(X)$. Lemma 7 states that $A \subseteq R^\bullet A$. To prove the reverse inclusion, we consider (making use of Prop. 5. of Lemma 3):

$$\begin{aligned}
R\uparrow A(x) &= \sup_{y \in X} T\big(A(y), R(y,x)\big) \\
&= \sup_{y \in X} T\big(R\downarrow C(y), R(y,x)\big) \\
&= \sup_{y \in X} T\big(R(y,x), \inf_{z \in X} \vec{T}(R(y,z), C(z))\big) \\
&\leq \sup_{y \in X} T\big(R(y,x), \vec{T}(R(y,x), C(x))\big) \\
&\leq C(x)
\end{aligned}$$

We have shown that $R\uparrow A \subseteq C$; therefore, by Lemma 5, we obtain

$$R^\bullet A = R\downarrow(R\uparrow A) \subseteq R\downarrow C = A,$$

which finally completes the proof. $\square$

The following fundamental theorem justifies to call R° the *opening operator of* R and to call $R^\bullet$ the *closure operator of* R.

Theorem 11. *For any fuzzy set $A \in \mathcal{F}(X)$, $R^\circ A$ is the largest R-open subset of A and $R^\bullet A$ is the smallest R-closed superset of A.*

Proof. Lemma 7 states that $R^\circ A \subseteq A$ and $R^\bullet A \supseteq A$.

Since $R^\circ A = R\uparrow(R\downarrow A)$, there exists a $B = R\downarrow A$ such that $R^\circ A = R\uparrow B$. Therefore, by Theorem 10, $R^\circ A$ is R-open. In an analogous way, we can prove that $R^\bullet A$ is R-closed (with $C = R\uparrow A$).

Now let us assume that B is an R-open subset of A. From the monotonicity property of the opening (cf. Lemma 8), we obtain

$$B = R^\circ B \subseteq R^\circ A.$$

Therefore, $R^\circ A$ must be the largest R-open subset of A .

Analogously, take an arbitrary R-closed fuzzy set with $C \supseteq A$ and the following holds:

$$C = R^\bullet C \supseteq R^\bullet A$$

Hence, there cannot exist any R-closed superset of A which is smaller than $R^\bullet A$. $\square$

Finally, we are able to prove another fundamental property of opening and closure operators—idempotency.

Corollary 12. *The following holds for any $A \in \mathcal{F}(X)$:*

$$R^\circ(R^\circ A) = R^\circ A \qquad (1)$$
$$R^\bullet(R^\bullet A) = R^\bullet A \qquad (2)$$

Proof. We know from Theorem 11 that $R^\circ(R^\circ A)$ is the largest subset of $R^\circ A$ which is R-open. Since $R^\circ A$ is R-open itself, the equality (1) must hold.

Analogously, the equality (2) can be deduced from Theorem 11. $\square$

4 Links to Existing Concepts

In the previous section, we have proposed generalized concepts of openness and closedness with respect to arbitrary fuzzy relations and the corresponding opening and closure operators. An important question, however, remained unclarified—whether the proposed concepts are useful and appropriate. We will try to answer this question by considering two special cases for which these concepts have existed already and which were not considered to be related so far—fuzzy preorderings and fuzzy mathematical morphology.

4.1 Openings and Closures with Respect to Fuzzy Preorderings

A concept of closedness with respect to a fuzzy relation appeared first in connection with fuzzy equivalence relations under the name "extensionality" [11, 13, 15]. The notion of extensionality and the corresponding opening and closure operators have turned out to be extremely helpful in practical terms, in particular when the analysis and interpretation of fuzzy partitions and controllers is concerned [10–13]. In [3, 4], these concepts were generalized to the non-symmetric case, i.e. to arbitrary fuzzy preorderings. Again, fruitful applications have demonstrated the richness and usefulness [2–4]. We will now show that the results from [3, 4] smoothly fit into the general framework.

Definition 13. A binary fuzzy relation $R \in \mathcal{F}(X^2)$ is called

1. *reflexive* if and only if $\forall x \in X : \; R(x,x) = 1$,
2. *symmetric* if and only if
$$\forall x,y \in X : \; R(x,y) = R(y,x),$$
3. *T-transitive* if and only if
$$\forall x,y,z \in X : \; T\big(R(x,y), R(y,z)\big) \leq R(x,z).$$

A reflexive and T-transitive fuzzy relation is called *fuzzy preordering* with respect to a t-norm T, short *T-preordering.* A symmetric T-preordering is called *fuzzy equivalence relation* with respect to T, short *T-equivalence.*

In previous studies [3, 4, 13, 15], closedness of a fuzzy set A with respect to a fuzzy preordering R was usually expressed by the property

$$\forall x,y \in X : \; T\big(A(x), R(x,y)\big) \leq A(y). \qquad (3)$$

We denote this property with *R-congruence* in the meantime. In the remaining section, let R be a fuzzy preordering with respect to some left-continuous t-norm T.

Nonchalantly speaking, the meaning of congruence is that, for any element $x \in A$, also all y are contained in A which are in relation to x.

The following theorem marks one of the cornerstones for investigating the relationship between R-congruence, R-closedness, and R-openness.

Theorem 14. [3, 4] *For any $A \in \mathcal{F}(X)$, $R{\downarrow}A$ is the largest R-congruent subset of A and $R{\uparrow}A$ is the smallest R-congruent superset.*

Theorem 14 already gives a hint that the two operators $R{\downarrow}$ and $R{\uparrow}$ can be considered as some kind of opening or closure operators, respectively, as long as a T-preordering is concerned.

As immediate consequences, we are able to deduce some fundamental properties.

Corollary 15. [3,4] *The following holds for any $A \in \mathcal{F}(X)$:*

1. *A is R-congruent if and only if $A = R{\downarrow}A$.*

2. *A is R-congruent if and only if $A = R{\uparrow}A$.*

3. *$R{\downarrow}(R{\downarrow}A) = R{\downarrow}A$*

4. *$R{\uparrow}(R{\uparrow}A) = R{\uparrow}A$*

5. *$R{\downarrow}(R{\uparrow}A) = R{\uparrow}A$*

6. *$R{\uparrow}(R{\downarrow}A) = R{\downarrow}A$*

In particular, Propositions 5. and 6. above state that, as long as we consider a T-preordering R, the two operators $R{\downarrow}$ and R° coincide, while, under the same conditions $R{\uparrow}$ and $R^{\bullet}$ are equal. This finally enables us to clarify the relationship between R-congruence and the general concepts of R-openness and R-closedness.

Theorem 16. *Provided that R is a T-preordering, the following three statements are equivalent for any fuzzy subset $A \in \mathcal{F}(X)$:*

(i) A is R-congruent.

(ii) A is R-open.

(iii) A is R-closed.

Proof. Immediate consequence of Corollary 15. □

Moreover, Theorem 14 and Corollary 15 guarantee that the opening and closure operators induced by the concept of R-congruence fit into the framework proposed in the previous section.

We conclude this section with a few simple examples which show the richness of properties that can be expressed by means of congruence, closedness, or openness of fuzzy preorderings:

1. A crisp set is closed with respect to a crisp equivalence relation if and only if it can be represented as the union of equivalence classes.

2. A crisp set is closed with respect to a crisp ordering if and only if it is an up-set.

3. A fuzzy set is closed with respect to a crisp ordering $\preceq$ if and only if its membership function is non-decreasing with respect to $\preceq$.

4.2 Fuzzy Mathematical Morphology Based on Triangular Norms

In a general setting, binary mathematical morphology is concerned with finding specific structures in binary images [18, 19]. Fuzzy mathematical morphology is an extension of binary mathematical morphology which allows treatment of gray-scale data by employing concepts from fuzzy set theory. One prominent approach among others is to use triangular norms and their residual implications in order to generalize the morphological operations [1, 5, 16]. Throughout the remaining section, assume that there exists a binary operation $+\colon X^2 \to X$ such that $(X, +)$ is an Abelian group, where we denote the neutral element with $\mathbf{0}$ and the inverse element of x with $-x$. For convenience, we abbreviate $x + (-y)$ with $x - y$.

The two key concepts of mathematical morphology are erosions and dilations. We adopt the notations of [8], whose generalization to the t-norm-based fuzzy case is rather straightforward [1, 5, 16].

Definition 17. Let $A, B \in \mathcal{F}(X)$ be two fuzzy sets. Then the *erosion* of A with respect to the structuring element B is defined as

$$(A \ominus B)(x) = \inf_{y \in X} \vec{T}\big(B(x-y), A(y)\big).$$

The *dilation* of A with respect to the structuring element B is defined as

$$(A \oplus B)(x) = \sup_{y \in X} T\big(A(y), B(y-x)\big).$$

The *opening* $A \circ B$ and the closing $A \bullet B$ of fuzzy set A with respect to the structuring element B are defined as

$$A \circ B = (A \ominus B) \oplus B,$$
$$A \bullet B = (A \oplus B) \ominus B.$$

Now let us clarify in which way fuzzy mathematical morphology, which does not involve fuzzy relations so far, relates to the concepts established earlier in this paper.

Definition 18. A binary fuzzy relation $R \in \mathcal{F}(X^2)$ is called *shift-invariant* if and only if the following holds for all $x, y, z \in X$:

$$R(x, y) = R(x + z, y + z)$$

Proposition 19. *A fuzzy relation $R \in \mathcal{F}(X^2)$ is shift-invariant if and only if there exists a fuzzy set $B \in \mathcal{F}(X)$ such that the following representation holds for all $x, y \in X$:*

$$R(x, y) = B(x - y) \tag{4}$$

Proof. First, we define $B(x) = R(x, \mathbf{0})$. Assuming that R is shift-invariant, we obtain

$$R(x, y) = R(x - y, y - y) = R(x - y, \mathbf{0}) = B(x - y).$$

Conversely, assume that Eq. (4) holds. Since $(X, +)$ is an Abelian group, this implies

$$\begin{aligned} R(x + z, y + z) &= B\big((x + z) - (y + z)\big) \\ &= B(x - y + z - z) \\ &= B(x - y) = R(x, y), \end{aligned}$$

which proves that R is shift-invariant. □

Proposition 19 demonstrates that shift-invariant fuzzy relations are uniquely characterized by a single fuzzy set—some kind of "structuring element". This representation can be utilized to show the major correspondence between the concepts defined for fuzzy relations and fuzzy mathematical morphology (see [17] for a similar argumentation).

Theorem 20. *Suppose that R is a shift-invariant fuzzy relation and that B is a fuzzy set such that representation (4) is satisfied. Then the following equalities hold for any $A \in \mathcal{F}(X)$:*

$$\begin{aligned} R{\downarrow}A &= A \ominus B \\ R{\uparrow}A &= A \oplus B \\ R^{\circ}A &= A \circ B \\ R^{\bullet}A &= A \bullet B \end{aligned}$$

Proof. Follows directly from the definitions. □

These correspondences allow to prove several important properties of fuzzy morphological operations.

Definition 21. Consider a fuzzy set $B \in \mathcal{F}(X)$. A fuzzy set $A \in \mathcal{F}(X)$ is called B-open if and only if $A \circ B = A$ holds. Correspondingly, A is called B-closed if and only if $A \bullet B = A$ holds (see also [5]).

Corollary 22. *The following assertions hold for all fuzzy sets $A, B \in \mathcal{F}(X)$:*

1. *$A \circ B$ is largest B-open fuzzy subset of A.*
2. *$A \bullet B$ is smallest B-closed fuzzy superset of A.*
3. *A is B-open if and only if there exists a fuzzy set $C \in \mathcal{F}(X)$ such that $A = C \oplus B$ holds.*
4. *A is B-closed if and only if there exists a fuzzy set $D \in \mathcal{F}(X)$ such that $A = D \ominus B$ holds.*
5. *$(A \circ B) \circ B = A \circ B$*
6. *$(A \bullet B) \bullet B = A \bullet B$*

Proof. If we define a fuzzy relation R by means of the structuring element B such that representation (4) holds, 1. and 2. can be proved by Theorem 11; 3. and 4. follow from Theorem 10, while 5. and 6. are immediate consequences of Corollary 12. □

Note that Propositions 3. and 4. of the previous theorem have already been proved directly in [5]. Equalities 5. and 6. have been proved in a slightly more general setting in the same paper, however, under the restriction that T has to be a continuous t-norm and that the structuring element has to have a finite range.

5 Concluding Remarks

This paper has been concerned with a general framework in which openness and closedness with respect to arbitrary fuzzy relations can be expressed. Within this framework, general opening and closure operators have been defined which are idempotent and fulfill all necessary extremal properties.

In the second part of the paper, we have demonstrated the richness of these notions by embedding existing concepts of opening and closure operators into the general framework. As two case studies, we have considered fuzzy preorderings and fuzzy morphological operations. In both cases, we have demonstrated full compatibility with the general setting. Moreover, we were able to deduce important nontrivial relationships from the general case.

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