Similarity-Based Fuzzy Orderings: A Comprehensive Overview

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Abstract

This contribution is intended to be a compendium of similarity-based fuzzy orderings—a generalization which has recently been discovered to be more appropriate for expressing vague order preferences than previously known concepts.

Keywords: fuzzy equivalence relation; fuzzy ordering.

1 Introduction

Models of preferences are essential components in decision making. In classical mathematics, relations, particularly *orderings*, are fundamental concepts for expressing preferences. In real-world situations, however, two-valued concepts are rarely rich enough to express the complexity of the way humans deal with preferences.

By allowing intermediate degrees of relationship, fuzzy relations provide much more freedom to express the subtle nuances of human preferences. Since their first appearance in 1971 [20], different variants of fuzzy orderings have become increasingly important in fuzzy preference modeling. However, the scientific community has witnessed the paradoxical development that the direct analogue to ordering relations (i.e. reflexive, transitive, and antisymmetric binary relations) has never played a significant role, although the corresponding crisp concept is of fundamental importance.

This paper is devoted to an overview of an alternative concept of fuzzy orderings which also takes the important relationship between fuzzy orderings and approximate similarity into account—in direct analogy to the fundamental connection between ordering and equivalence relations in the classical two-valued case. After providing the reader with the most important preliminaries (Section 2), we will briefly motivate and introduce the generalized approach (Section 3). In Section 4, three examples will give the reader an impression of the richness and appropriateness of the concept. Finally, Section 5 is devoted to a comprehensive survey of construction and representation results.

2 Preliminaries

We will deal with fuzzy relations in a rather classical sense here, i.e. the unit interval $[0,1]$ is considered as the standard domain of truth values. A *fuzzy set* on a crisp non-empty domain X is uniquely characterized by a $X \to [0,1]$ membership function. We will use uppercase letters for fuzzy sets and their membership functions synonymously; the same for *binary fuzzy relations* on a non-empty crisp domain X which are supposed to be fuzzy sets on $X \times X$.

In this paper, logical conjunctions will solely be modeled by *triangular norms* (short t-norms), i.e. associative, commutative, and non-decreasing binary operations on the unit interval ($[0,1]^2 \to [0,1]$ mappings) which have 1 as neutral element. A t-norm T is called *left-continuous* if and only if both partial mappings $T(x,.)$ and $T(.,x)$ are left-continuous.

Definition 1. Consider a t-norm T. A binary fuzzy relation $R : X^2 \to [0,1]$ is called

1. *reflexive* if and only if, for all $x \in X$, $R(x,x) = 1$,
2. *symmetric* if and only if, for all $x, y \in X$, $R(x,y) = R(y,x)$,
3. T*-antisymmetric* if and only if, for all $x, y \in X$, $x \neq y \Rightarrow T\big(R(x,y), R(y,x)\big) = 0$,
4. T*-transitive* if and only if, for all $x, y, z \in X$, $T\big(R(x,y), R(y,z)\big) \leq R(x,z)$,
5. *strongly complete* or, synonymously, *strongly linear* if and only if, for all $x, y \in X$, $\max\big(R(x,y), R(y,x)\big) = 1$.

Definition 2. A reflexive and T-transitive fuzzy relation is called *fuzzy preordering* with respect to a t-norm T, short T*-preordering.* A T-preordering which is, in addition, symmetric is called *fuzzy equivalence relation* with respect to T, short T*-equivalence.*

For intersecting T-transitive fuzzy relations, the concept of domination between t-norms is of vital importance [7, 15].

Definition 3. A t-norm T_1 is said to *dominate* another t-norm T_2 if and only if, for any quadruple $(x, y, u, v) \in [0,1]^4$, the following holds:

$$T_1\big(T_2(x,y), T_2(u,v)\big) \geq T_2\big(T_1(x,u), T_1(y,v)\big)$$

Lemma 4. [7] *Consider two t-norms T_1 and T_2. The T_2-intersection of any two arbitrary T_1-transitive fuzzy relations is T_1-transitive if and only if T_2 dominates T_1.*

3 Fuzzy Orderings

The most obvious idea to define fuzzy orderings is, of course, to demand the three straightforward generalizations of the classical axioms reflexivity, antisymmetry, and transitivity (cf. Definition 1). The first definition of that type already appears in [20] under the slightly misleading name *fuzzy partial ordering*, where only the so-called minimum t-norm $T_{\mathbf{M}}(x,y) = \min(x,y)$ was considered. Here, we give the more general definition admitting an arbitrary t-norm.

Definition 5. Let T be an arbitrary t-norm. A reflexive, T-antisymmetric, and T-transitive binary fuzzy relation is called *fuzzy ordering* with respect to the t-norm T, for brevity T*-ordering.*

Doubts that T-antisymmetry could be too strong a requirement have already appeared rather early and have motivated several researchers to propose generalizations—some with less axioms, some with weakened axioms [9, 17, 20]. Fuzzy orderings with a link to an underlying concept of indistinguishability of approximate similarity appeared first in a paper by U. Höhle and N. Blanchard [12]. The introduction of this paper contains the following remarkable statement:

> *"In opposition to Zadeh's, our point of view is that an axiom of antisymmetry without reference to a concept of equality is meaningless."*

However, the authors' argumentation is rather based on formal category theory-related considerations than on practical aspects of fuzzy set theory and preference modeling. For this reason and since the paper, as a whole, was strictly based on a category-theoretic setting, this fundamental contribution, unfortunately, remained unrecognized in the fuzzy set and decision making communities.

In more recent investigations [4, 5], the same idea was re-vitalized in the setting of fuzzy set theory—also providing case studies which demonstrate the appropriateness of the similarity-based generalization.

Definition 6. Consider a binary fuzzy relation $L : X^2 \to [0,1]$. L is called *fuzzy ordering* with respect to a t-norm T and a T-equivalence E (on the same domain X), for brevity T*-*E*-ordering,* if and only if it is T-transitive and additionally fulfills the following two axioms:

1. E*-reflexivity*, i.e. for all $x, y \in X$, $E(x,y) \leq L(x,y)$
2. T*-*E*-antisymmetry*, i.e. for all $x, y \in X$, $T\big(L(x,y), L(y,x)\big) \leq E(x,y)$

It is easy to verify that Definition 6 is consistent with the previous definition of T-orderings (cf. Def. 5) if E is replaced by the crisp equality.

4 Examples

In order to give the reader an impression which kinds of relations fuzzy orderings are and what they may be used for, we give three simple examples. The first two are more of theoretical interest, while the third one has a clear application background.

4.1 Implications as Orderings

Consider an arbitrary left-continuous t-norm T. Then its so-called residual implication [10, 11, 15], defined as

$$\vec{T}(x,y) = \sup\{u \in [0,1] \mid T(u,x) \leq y\},$$

is a strongly linear fuzzy ordering on the unit interval with respect to T and the corresponding biimplication

$$\overleftrightarrow{T}(x,y) = T\big(\vec{T}(x,y), \vec{T}(y,x)\big).$$

This correspondence can be considered as a fuzzy analogue to the well-known fact from classical logic that the relation

$$\varphi \lesssim \psi \iff (\varphi \to \psi \text{ is a tautology}),$$

is an ordering, if φ and ψ are formulas and if we always consider two formulas as equal if their evaluations coincide for all interpretations.

Note that $\vec{T}$ can never fulfill T-antisymmetry, regardless of the choice of T.

4.2 Fuzzy Inclusions

If we fix a left-continuous t-norm T, the fuzzy inclusion relation [1, 10]

$$\mathrm{INCL}_T(A,B) = \inf_{x\in X} \vec{T}\big(A(x), B(x)\big)$$

defines a fuzzy ordering on the fuzzy power set $\mathcal{F}(X)$ with respect to T and

$$\mathrm{SIM}_T(A,B) = \inf_{x\in X} \overleftrightarrow{T}\big(A(x), B(x)\big)$$

which is a well-known mapping for measuring the similarity of fuzzy sets, at least if T equals the Łukasiewicz t-norm [16, 19]

$$T_{\mathbf{L}}(x,y) = \max(x+y-1, 0).$$

It is worth to mention that, for any left-continuous t-norm T, INCL_T is not T-antisymmetric.

4.3 Linear Orderings with Imprecision

The fuzzy relation

$$L(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ \max(1-x+y, 0) & \text{otherwise} \end{cases}$$

is a fuzzy ordering on the real numbers $\mathbb{R}$ with respect to the Łukasiewicz t-norm $T_{\mathbf{L}}$ and the following well-known $T_{\mathbf{L}}$-equivalence [6, 16]:

$$E(x,y) = \max(1 - |x-y|, 0)$$

This example corresponds to a situation familiar from everyday life—even if a clear crisp linear ordering is known, there is a certain tolerance for indistinguishability which is taken into account even when ordering is concerned. Let us consider the following two examples:

- Humans usually do not interpret the relation "at least as tall as" in the strict sense $\mathrm{height}(x) \leq \mathrm{height}(y)$, they also take into account that the difference between some heights, e.g. 179.9cm and 180.0cm, is almost negligible.
- Suppose that somebody sends a query to an online tourist information system asking for a hotel room which should fulfill some quality criteria, but which should not be more expensive than € 70.00 per night. In any case, it seems inappropriate not to offer him/her a room which costs € 70.10, provided that all other search criteria are fulfilled.

This interplay between crisp linear orderings and fuzzy equivalence relations is also a typical phenomenon in fuzzy control [13, 14]. The similarity-based approach to fuzzy orderings, hereby, opens a new direction of applications of fuzzy relations in fuzzy control and approximate reasoning [2, 8].

Note that such a kind of "relaxed" linear ordering cannot be modeled by a T-ordering. It is immediate to see that there exists no non-trivial extension of a crisp linear ordering which fulfills the axioms of Definition 5 [4, 5].

5 Constructions and Representations

In this section, we give a comprehensive overview of different representations and constructions.

For detailed proofs, we refer to already published material [3, 4, 5].

5.1 Implicit Factorization

It is easy to prove that the symmetric kernel of a crisp preordering is an equivalence relation. A fuzzy analogue has been proved in [18]. The following result shows, one step further, how to construct T-equivalences such that a given T-preordering can be considered as a fuzzy ordering.

Theorem 7. *Suppose L to be a T-preordering and $\tilde{T}$ to be a t-norm which dominates T. Then L is a fuzzy ordering with respect to T and*

$$E(x,y) = \tilde{T}\big(L(x,y), L(y,x)\big).$$

Since it is trivial to prove that any t-norm T dominates itself and that the minimum t-norm $T_{\mathbf{M}}$ dominates any other t-norm T [7], the assertion of Theorem 7 also holds for $\tilde{T} = T$ and $\tilde{T} = T_{\mathbf{M}}$, and we obtain unique upper and lower bounds for the underlying fuzzy equivalence relation.

Corollary 8. *A T-preordering L is a fuzzy ordering with respect to T and a T-equivalence E if and only if, for all $x, y \in X$,*

$$\begin{aligned} T\big(L(x,y), L(y,x)\big) &\leq E(x,y) \\ &\leq \min\big(L(x,y), L(y,x)\big). \end{aligned}$$

At first glance, Theorem 7 seems to indicate that T-E-antisymmetry is a superfluous axiom, since any T-preordering can be interpreted as a T-E-ordering (for some E) anyway. However, the existence of such an E only means that L can be considered as a reasonable concept of ordering if we may consider E as an appropriate concept of approximate similarity in the given environment—otherwise the relation E is of no practical use and its introduction is purely artificial. In many practical situations, the underlying context of indistinguishability E is given in advance anyway.

In any case, we face the same situation in the crisp case—any crisp preordering can be considered an ordering if we factorize with respect to its symmetric kernel. Although factorization is not an explicit element of Theorem 7 and Corollary 8, they together constitute a perfect analogue to factorization argument in the crisp case. Moreover, we should not forget that such kinds of "implicit factorization" even occur in the crisp case when we often nonchalantly say "equal" while meaning "equivalent" (e.g. like in Subsection 4.1)

5.2 Intersections and Cartesian Products

The following theorem gives a sufficient condition under which the intersection of two fuzzy orderings again yields a fuzzy ordering.

Theorem 9. *Suppose that L_1 is a T-E_1-ordering on X and L_2 is a T-E_2-ordering on X. If $\tilde{T}$ is a t-norm which dominates T, then*

$$L(x,y) = \tilde{T}(L_1(x,y), L_2(x,y))$$

is a T-E-ordering with

$$E(x,y) = \tilde{T}(E_1(x,y), E_2(x,y)).$$

By induction, Theorem 9 can be generalized to an intersection of any finite number of fuzzy orderings. If the minimum t-norm $T_{\mathbf{M}}$ is considered, this even works for the infinite case.

Corollary 10. *Let $(L_i)_{i \in I}$ and $(E_i)_{i \in I}$ be two (possibly infinite) families of fuzzy relations on X such that each L_i is a T-E_i-ordering. Then*

$$L(x,y) = \inf_{i \in I} L_i(x,y)$$

is a T-E-ordering with

$$E(x,y) = \inf_{i \in I} E_i(x,y).$$

Since Cartesian products are nothing else than intersections of cylindrical extensions, we obtain a basic procedure how to construct fuzzy orderings on product spaces provided that we are given fuzzy orderings in each component.

Theorem 11. *Let us consider a finite family of non-empty crisp sets $(X_1, \ldots, X_n)$, an arbitrary t-norm T, and two families of fuzzy relations $(L_1, \ldots, L_n)$ and $(E_1, \ldots, E_n)$ such that, for all $i \in \{1, \ldots, n\}$, E_i is a T-equivalence on X_i and L_i is a T-E_i-ordering on X_i. If a t-norm $\tilde{T}$ dominates T, the mapping*

$$\begin{aligned} \tilde{L}: \quad (X_1 \times \cdots \times X_n)^2 &\to [0,1] \\ \big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) &\mapsto \mathop{\tilde{T}}_{1 \leq i \leq n} L_i(x_i, y_i) \end{aligned}$$

is a fuzzy ordering with respect to T and the fuzzy equivalence relation

$$\tilde{E}\big((x_1,\ldots,x_n),(y_1,\ldots,y_n)\big) = \mathop{\tilde{T}}_{1\leq i\leq n} E_i(x_i,y_i).$$

Almost needless to mention, Theorem 11 can also be generalized to infinite Cartesian products if the minimum t-norm is considered (as a consequence of Corollary 10).

Note that it still remains an open problem how to define fuzzy orderings on product spaces by means of a kind of lexicographic composition.

5.3 Fuzzifications of Crisp Orderings

In Subsection 4.3, we have discussed the general motivation behind, what we called, *linear orderings with imprecision*, i.e. fuzzy orderings which fuzzify a crisp linear ordering. We have already mentioned a simple example; now it is time to introduce a general representation.

Definition 12. Let $\lesssim$ be a crisp ordering on X and let E be a fuzzy equivalence relation on X. E is called *compatible with* $\lesssim$, if and only if the following implication holds for all $x, y, z \in X$:

$$x \lesssim y \lesssim z \Longrightarrow E(x,z) \leq \min\big(E(x,y),E(y,z)\big)$$

This property can be interpreted as follows: The two outer elements of a three-element chain are at least as distinguishable as any two inner elements.

Theorem 13. *Consider a fuzzy relation L on a domain X and a T-equivalence E. Then the following two statements are equivalent:*

(i) L is a strongly linear T-E-ordering.

(ii) There exists a linear ordering $\lesssim$ the relation E is compatible with such that L can be represented as follows:

$$L(x,y) = \begin{cases} 1 & \textit{if } x \lesssim y \\ E(x,y) & \textit{otherwise} \end{cases}$$

Theorem 13 states that strongly linear fuzzy orderings are uniquely characterized as fuzzifications of crisp linear orderings, where the fuzzy component can be attributed to a fuzzy equivalence relation. By means of the elaborate theory of fuzzy equivalence relations [6], Theorem 13 can also be used to construct strongly linear fuzzy orderings from pseudo-metrics (see [3] for more details).

5.4 Fuzzy Orderings Generated by Families of Fuzzy Sets

Since any T-E-ordering is a T-preordering, the well-known representation theorem holds which we cite next. For the whole subsection, let us assume that T denotes a left-continuous t-norm.

Theorem 14. [18] *Provided that L is a binary fuzzy relation on a domain X, the following two statements are equivalent:*

(i) L is a T-preordering.

(ii) There exists a family of fuzzy sets $(A_i)_{i\in I}$ on X such that the following representation holds:

$$L(x,y) = \inf_{i\in I} \vec{T}\big(A_i(x),A_i(y)\big) \qquad (1)$$

An analogous theorem holds for fuzzy equivalence relations.

Theorem 15. [18] *Given a binary fuzzy relation E on a domain X, the following two statements are equivalent:*

(i) E is a T-equivalence.

(ii) There exists a family of fuzzy sets $(A_i)_{i\in I}$ on X such that the following representation holds:

$$E(x,y) = \inf_{i\in I} \overleftrightarrow{T}\big(A_i(x),A_i(y)\big) \qquad (2)$$

As we have seen in Subsection 5.1, T-E-orderings are nothing else than T-preorderings with a special interaction with a given T-equivalence E. The question arises whether, for a given T-E-ordering L, the fuzzy relations L and E can be generated from the same family of fuzzy sets. In general, the answer is negative. The following theorem shows that at least one of the two directions holds.

Theorem 16. *Let $(A_i)_{i\in I}$ be a family of fuzzy sets on a domain X. Then L in Eq. (1) defines a T-E-ordering, where E is a T-equivalence defined like in Eq. (2).*

The next theorem gives a unique representation, however, only for a subclass which fulfills a slightly stricter antisymmetry criterion.

Theorem 17. *Consider two binary fuzzy relations E and L on a domain X. Then the following two statements are equivalent:*

(i) E is a T-equivalence and L is a T-E-ordering which fulfills the following property (for all $x, y \in X$):

$$\min\bigl(L(x,y), L(y,x)\bigr) \leq E(x,y) \qquad (3)$$

(ii) There exists a family of fuzzy subsets $(A_i)_{i\in I}$ of X such that L can be represented as in (1) and such that E can be represented like in (2).

Note that (3) is automatically fulfilled if $T = T_{\mathbf{M}}$ or if L is strongly linear.

5.5 Inclusion-Based Representations

Very similar proof techniques as in the previous subsection can be employed to show that T-preorderings, T-equivalences, and, again most importantly for us, T-E-orderings can be reduced to the fundamental set comparisons INCL_T and SIM_T, where T again denotes a left-continuous t-norm.

Theorem 18. *Provided that L is a binary fuzzy relation on a domain X, the following two statements are equivalent:*

(i) L is a T-preordering.

(ii) There exists an embedding mapping $\varphi : X \to \mathcal{F}(X)$ such that the following representation holds:

$$L(x,y) = \mathrm{INCL}_T\bigl(\varphi(x), \varphi(y)\bigr) \qquad (4)$$

Again, an analogous correspondence holds for fuzzy equivalence relations.

Theorem 19. *Given a binary fuzzy relation E on a domain X, the following two statements are equivalent:*

(i) E is a T-equivalence.

(ii) There exists an embedding mapping $\varphi : X \to \mathcal{F}(X)$ such that the following representation holds:

$$E(x,y) = \mathrm{SIM}_T\bigl(\varphi(x), \varphi(y)\bigr) \qquad (5)$$

In perfect analogy to Subsection 5.4, a construction based on the fuzzy relations INCL_T and SIM_T can be formulated.

Theorem 20. *Consider a mapping $\varphi : X \to \mathcal{F}(X)$. Then L in Eq. (4) defines a T-E-ordering, where E is a T-equivalence defined as in Eq. (5).*

In the same way as above, the reverse of Theorem 20 only holds for the subclass satisfying the stronger antisymmetry axiom (3).

Theorem 21. *Let E and L be two binary fuzzy relations on a domain X. Then the following two statements are equivalent:*

(i) E is a T-equivalence and L is a T-E-ordering that fulfills property (3).

(ii) There exists an embedding mapping $\varphi : X \to \mathcal{F}(X)$ such that L can be represented like in (5) and such that E can be represented as in Eq. (4).

6 Concluding Remarks

This paper has been devoted to a new fundamental model for expressing order preferences in vague environments. We have elucidated the motivation, historical development, and basic properties. By means of examples, representations, and constructions, we have tried to demonstrate the richness and soundness of this concept.

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References

[1] W. Bandler and L. Kohout. Fuzzy power sets and fuzzy implication operators. *Fuzzy Sets and Systems*, 4:183–190, 1980.

[2] U. Bodenhofer. The construction of ordering-based modifiers. In G. Brewka, R. Der, S. Gottwald, and A. Schierwagen, editors, *Fuzzy-Neuro Systems '99*, pages 55–62. Leipziger Universitätsverlag, 1999.

[3] U. Bodenhofer. Representations and constructions of strongly linear fuzzy orderings. In *Proc. EUSFLAT-ESTYLF'99*, pages 215–218, 1999.

[4] U. Bodenhofer. *A Similarity-Based Generalization of Fuzzy Orderings*, volume C 26 of *Schriftenreihe der Johannes-Kepler-Universität Linz*. Universitätsverlag Rudolf Trauner, 1999.

[5] U. Bodenhofer. A similarity-based generalization of fuzzy orderings preserving the classical axioms. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 8(5):593–610, 2000.

[6] B. De Baets and R. Mesiar. Pseudo-metrics and T-equivalences. *J. Fuzzy Math.*, 5(2):471–481, 1997.

[7] B. De Baets and R. Mesiar. T-partitions. *Fuzzy Sets and Systems*, 97:211–223, 1998.

[8] M. De Cock, U. Bodenhofer, and E. E. Kerre. Modelling linguistic expressions using fuzzy relations. In *Proc. IIZUKA2000*, pages 353–360, Iizuka, October 2000.

[9] J. Fodor and M. Roubens. *Fuzzy Preference Modelling and Multicriteria Decision Support*. Kluwer Academic Publishers, Dordrecht, 1994.

[10] S. Gottwald. *Fuzzy Sets and Fuzzy Logic*. Vieweg, Braunschweig, 1993.

[11] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer Academic Publishers, Dordrecht, 1998.

[12] U. Höhle and N. Blanchard. Partial ordering in L-underdeterminate sets. *Inform. Sci.*, 35:133–144, 1985.

[13] F. Klawonn, J. Gebhardt, and R. Kruse. Fuzzy control on the basis of equality relations—with an example from idle speed control. *IEEE Trans. Fuzzy Systems*, 3:336–356, 1993.

[14] F. Klawonn and R. Kruse. Equality relations as a basis for fuzzy control. *Fuzzy Sets and Systems*, 54(2):147–156, 1993.

[15] E. P. Klement, R. Mesiar, and E. Pap. *Triangular Norms*, volume 8 of *Trends in Logic*. Kluwer Academic Publishers, Dordrecht, 2000.

[16] R. Kruse, J. Gebhardt, and F. Klawonn. *Foundations of Fuzzy Systems*. John Wiley & Sons, New York, 1994.

[17] S. V. Ovchinnikov. Similarity relations, fuzzy partitions, and fuzzy orderings. *Fuzzy Sets and Systems*, 40(1):107–126, 1991.

[18] L. Valverde. On the structure of F-indistinguishability operators. *Fuzzy Sets and Systems*, 17(3):313–328, 1985.

[19] X. Wang, B. De Baets, and E. E. Kerre. A comparative study of similarity measures. *Fuzzy Sets and Systems*, 73:259–268, 1995.

[20] L. A. Zadeh. Similarity relations and fuzzy orderings. *Inform. Sci.*, 3:177–200, 1971.